We now come back to the free SE $i\partial_t \psi(t,x) = -\frac{1}{2}\Delta_x \psi(t,x)$

Formally we solve this by applying $\mathcal{F}$: $i\partial_t \hat{\psi}(t,k) = -\frac{1}{2}(\mathcal{F}\Delta_x\psi)(t,k) \overset{\text{Lemma 2.11}}{=} \frac{1}{2}k^2\hat{\psi}(t,k)$

$\Rightarrow \hat{\psi}(t,k) = e^{-i\frac{k^2}{2}t}\hat{\psi}(0,k)$ unique global solution

$\Rightarrow \psi(t,x) = \left(\mathcal{F}^{-1}e^{-i\frac{k^2}{2}t}\mathcal{F}\psi_0\right)(x)$, with $\psi_0(x) = \psi(0,x)$ the initial condition

Theorem 2.16: Solution to free SE in $\mathcal{S}$

Let $\psi_0 \in \mathcal{S}(\mathbb{R}^d)$. Then the unique $\underbrace{\text{global}}_{\text{for all } t \text{ (as opposed to "local" = for some finite time interval)}}$ solution $\psi \in C^\infty(\mathbb{R}_t, \mathcal{S}(\mathbb{R}^d))$ to the free SE with $\psi(0,x) = \psi_0(x)$ is, for $t \neq 0$,

$$\psi(t,x) = \left(\mathcal{F}^{-1}e^{-i\frac{k^2}{2}t}\mathcal{F}\psi_0\right)(x) = (2\pi i t)^{-\frac{d}{2}} \int_{\mathbb{R}^d} e^{i\frac{(x-y)^2}{2t}}\psi_0(y)\,dy.$$

Furthermore, $\|\psi(t,\cdot)\|_{L^2(\mathbb{R}^d)} = \|\psi_0\|_{L^2(\mathbb{R}^d)}$ $\forall t \in \mathbb{R}$.

Important note: What does $\psi \in C^\infty(\mathbb{R}_t, \mathcal{S}(\mathbb{R}^d))$ mean?

First, ψ is a map from $\mathbb{R}$ to $\mathcal{S}(\mathbb{R}^d)$, i.e., for fixed t, $\psi(t,x)$ as a function of x lies in $\mathcal{S}$.

Second, the map $\psi: \mathbb{R}_t \to \mathcal{S}(\mathbb{R}^d)$ is ∞-often differentiable, i.e.,

$\frac{\psi(t+h,\cdot) - \psi(t,\cdot)}{h} \xrightarrow[\text{in } \mathcal{S}]{h\to 0} \dot{\psi}(t)$ for some $\dot{\psi}(t) \in \mathcal{S}$.

Proof: The formula $\psi(t,x) = \left(\mathcal{F}^{-1} e^{-i\frac{k^2}{2}t} \mathcal{F}\psi_0\right)(x) = (2\pi i t)^{-\frac{d}{2}} \int_{\mathbb{R}^d} e^{i\frac{|x-y|^2}{2t}} \psi_0(y)\,dy$

can be checked by direct computation (use Fourier transform of Gaussian).

Next: Let us show that $t \mapsto \psi(t,\cdot)$ is once differentiable, then

$\psi \in C^\infty(\mathbb{R}, \mathcal{S})$ follows by repeating the argument.

Guess: derivative is $\dot{\psi}(t,x) = -i\left(\mathcal{F}^{-1}\frac{k^2}{2} e^{-i\frac{k^2}{2}t} \mathcal{F}\psi_0\right)(x)$, which we know is in $\mathcal{S}(\mathbb{R}^d)$.

To show: $\lim_{h\to 0} \left\| \frac{\psi(t+h)-\psi(t)}{h} - \dot{\psi}(t) \right\|_{\alpha,\beta} = 0 \quad \forall \alpha,\beta \in \mathbb{N}_0^d$

By continuity of $\mathcal{F}$ (Lemma 2.11), this is equivalent to

$$\underbrace{\lim_{h\to 0} \left\| \mathcal{F}\left(\frac{\psi(t+h)-\psi(t)}{h} - \dot{\psi}(t) \right) \right\|_{\alpha,\beta}} = 0 \quad \forall \alpha,\beta \in \mathbb{N}_0^d$$

$$= \lim_{h\to 0} \left\| \frac{\hat{\psi}(t+h)-\hat{\psi}(t)}{h} - \hat{\dot{\psi}}(t) \right\|_{\alpha,\beta}$$

$$= \lim_{h\to 0} \sup_{k\in\mathbb{R}^d} \left| \underbrace{k^\alpha \partial_k^\beta \left(\frac{e^{-i\frac{k^2}{2}(t+h)} - e^{-i\frac{k^2}{2}t}}{h} + i\frac{k^2}{2} e^{-i\frac{k^2}{2}t} \right)(\mathcal{F}\psi_0)(k)} \right| = 0,$$

$\leq h\, f(h,k)$, with $\lim_{h\to 0} f(h,k) \leq C \;\forall k\in\mathbb{R}^d$ ($e^{-i\frac{k^2}{2}t}$ smooth, $\hat{\psi}_0 \in \mathcal{S}$)

and $\sup_{k\in\mathbb{R}^d} f(h,k) \leq C \;\forall h\in\mathbb{R}$ ($\hat{\psi}_0 \in \mathcal{S}$)

$= 0$

We compute furthermore:

$$\|\psi(t,\cdot)\|_{L^2}^2 = \int |\psi(t,x)|^2 dx = \int \left|\left(\mathcal{F}^{-1} e^{-i\frac{k^2}{2}t}\mathcal{F}\psi_0\right)(x)\right|^2 dx$$

$$\overset{\text{Plancherel (2.14)}}{=} \int \left|e^{-i\frac{k^2}{2}t}\mathcal{F}\psi_0(x)\right|^2 dx$$

$$= \int |\mathcal{F}\psi_0(x)|^2 dx$$

$$\overset{\text{Plancherel (2.14)}}{=} \int |\psi_0(x)|^2 dx = \|\psi(0,\cdot)\|_{L^2}^2 .$$

$\square$

Note: $\|\psi(t,\cdot)\|_\infty = \sup_{x\in\mathbb{R}^d} |\psi(t,x)| = \sup_{x\in\mathbb{R}^d} \left| (2\pi i t)^{-\frac{d}{2}} \int e^{i\frac{|x-y|^2}{2t}} \psi_0(y)\, dy \right|$

$$\leq (2\pi t)^{-\frac{d}{2}} \|\psi_0\|_{L^1} \xrightarrow{t\to\infty} 0$$

$\Rightarrow$ wave functions spread: