We want to define multiplication operators $\psi(x) \mapsto f(x)\psi(x)$ as continuous maps on $\mathcal{S}$, as we did with $e^{-i\frac{k^2}{2}t}\hat{\psi}_0$. For that, f cannot be too wild; an appropriate space is:

Definition 2.18: The space of smooth polynomially bounded functions is

$$C^\infty_{pol}(\mathbb{R}^d) := \left\{ f \in C^\infty(\mathbb{R}^d) : \forall \alpha \in \mathbb{N}_0^d \; \exists n_\alpha \in \mathbb{N} \text{ and } C_\alpha < \infty \text{ s.t. } |\partial_x^\alpha f(x)| \leq C_\alpha (1+|x|^2)^{\frac{n_\alpha}{2}} \right\}$$

Note:
- a common notation is: $(1+|x|^2)^{\frac{1}{2}} =: \langle x \rangle$
- e.g., all polynomials $\in C^\infty_{pol}$, $e^{ikx} \in C^\infty_{pol}$, $e^x \notin C^\infty_{pol}$

Then indeed:

Lemma: For $f \in C^\infty_{pol}(\mathbb{R}^d)$, the multiplication operator $M_f : \mathcal{S} \to \mathcal{S}$, $\psi(x) \mapsto f(x)\psi(x)$ is continuous.

Proof: clear: if $\|\psi_n - \psi\|_{\alpha,\beta} \xrightarrow{n\to\infty} 0 \;\forall \alpha, \beta$, then also

$$\|M_f(\psi_n - \psi)\|_{\alpha,\beta} = \sup_{x \in \mathbb{R}^d} |x^\alpha \partial_x^\beta (f(x)(\psi_n(x) - \psi(x)))| \xrightarrow{n\to\infty} 0 \qquad \square$$

The solution to the free SE can be written as $\mathcal{F}^{-1} e^{-i\frac{k^2}{2}t} \mathcal{F} \psi_0 = \mathcal{F}^{-1} M_f \mathcal{F} \psi_0$ for $f(k) = e^{-i\frac{k^2}{2}t}$. Since multiplication in Fourier space = derivatives in x-space, we introduce the following notation for $\mathcal{F}^{-1} M_f \mathcal{F}$:

Definition 2.19:

For $f \in C^\infty_{pol}(\mathbb{R}^d)$ we define the pseudo-differential operator

$$f(-i\nabla): \mathcal{S}(\mathbb{R}^d) \to \mathcal{S}(\mathbb{R}^d),\ \psi(x) \longmapsto \left(f(-i\nabla_x)\psi\right)(x) = \left(\mathcal{F}^{-1} M_f \mathcal{F}\psi\right)(x) = \left(\mathcal{F}^{-1} f(k) \mathcal{F}\psi\right)(x)$$

Note:
- $f(-i\nabla)$ continuous, since $M_f, \mathcal{F}, \mathcal{F}^{-1}$ continuous
- $f(k) = k^\alpha \Rightarrow f(-i\nabla) = (-i)^{|\alpha|}\partial_x^\alpha$ is the usual differential operator
- Example: semi-relativistic or pseudo-relativistic Schrödinger equation:

$$i\partial_t \psi(t,x) = \underbrace{\sqrt{1-\Delta}}_{\text{pseudo-differential operator}}\psi(t,x)$$

Examples:

- translation operator: for $a \in \mathbb{R}^d$, let $T_a(k) = e^{-iak} \Rightarrow T_a \in C^\infty_{pol}$

$\Rightarrow$ for $\psi \in \mathcal{S}$, we find
$$\left(T_a(-i\nabla)\psi\right)(x) = (2\pi)^{-\frac{d}{2}} \int e^{ikx} e^{-iak}\hat{\psi}(k)\,dk$$
$$= (2\pi)^{-\frac{d}{2}} \int e^{ik(x-a)}\hat{\psi}(k)\,dk$$
$$= \psi(x-a)$$

- free propagator: $P_f(k) = e^{-i\frac{k^2}{2}t} \Rightarrow P_f \in C^\infty_{pol}$

$\Rightarrow$ solution to free Schrödinger equation is
$$\psi(t,x) = \left(\mathcal{F}^{-1} e^{-i\frac{k^2}{2}t}\mathcal{F}\psi_0\right)(x)$$
$$= \left(P_f(t,-i\nabla)\psi_0\right)(x)$$

Thus, we found: $\psi(t) = e^{-i\frac{(-\Delta)}{2}t}\psi(0)$

- heat equation: $\partial_t f(t,x) = \frac{1}{2}\Delta_x f(t,x)$

$\Rightarrow W(t,k) = e^{-\frac{k^2}{2}t} \in C^\infty_{pol}$ for $t \geq 0$

$\Rightarrow$ for $f(0,\cdot) = f_0 \in S$, $t > 0$, we have $f(t) = e^{\frac{1}{2}\Delta t} f_0 = W(t,-i\nabla) f_0$

Lastly, $\mathcal{F}^{-1} M_f \mathcal{F} \psi_0 = \mathcal{F}^{-1}(f(k)\hat{\psi}_0(k))$, so we want to know about the (inverse) Fourier transform of a product.

Definition 2.22:

The convolution of $f \in S$ and $g \in S$ is $(f * g)(x) := \int_{\mathbb{R}^d} f(x-y) g(y) dy$.

Lemma 2.23: For $f, g, h \in S$ we have

a) $(f*g)*h = f*(g*h)$ and $f*g = g*f$;

b) the map $S \to S$, $g \mapsto f*g$ is continuous;

c) $\widehat{f*g} = (2\pi)^{\frac{d}{2}} \hat{f} \cdot \hat{g}$ and $\widehat{f g} = (2\pi)^{-\frac{d}{2}} \hat{f} * \hat{g}$,

in particular $g(-i\nabla) f = \mathcal{F}^{-1} M_g \mathcal{F} f = \mathcal{F}^{-1} g \hat{f} = (2\pi)^{-\frac{d}{2}} \check{g} * f$.

Proof: • a) and c) are direct calculations

• then b) follows since $f * g = (2\pi)^{\frac{d}{2}} \mathcal{F}^{-1} \hat{f}\hat{g}$, i.e., composition of continuous maps $\square$

Example: heat equation: $f(t,x) = W(t,-i\nabla) f(0,x)$, $W(t,k) = e^{-\frac{k^2}{2}t}$

$$= (2\pi)^{-\frac{d}{2}} \left((\mathcal{F}^{-1} W_t) * f_0 \right)(x)$$

with heat kernel $G(t,x) = (2\pi)^{-\frac{d}{2}} (\mathcal{F}^{-1} W)(t,x) = (2\pi t)^{-\frac{d}{2}} e^{-\frac{x^2}{2t}}$ we find

$$f(t,x) = \left(G(t) * f_0\right)(x) = (2\pi t)^{-\frac{d}{2}} \int e^{-\frac{(x-y)^2}{2t}} f_0(y)\, dy$$

To summarize: The solution to the free SE is

$$\psi_0(t) = \mathcal{F}^{-1} M_{P_f} \mathcal{F} \psi_0 = e^{-i\left(\frac{-\Delta}{2}\right)t} \psi_0 = G(t) * \psi_0 \quad \text{, with } P_f(k) = e^{-i\frac{k^2}{2}t} \text{, } G(t,x) = (2\pi i t)^{-\frac{d}{2}} e^{\frac{i(x-y)^2}{2t}}$$