2.2 Tempered Distributions

Definition 2.27:

Let V be a topological vector space over a field $\mathbb{F}$ (here usually $\mathbb{F}=\mathbb{C}$).
Then the dual space V' is the space of all continuous linear maps $V \to \mathbb{F}$.

For $f \in V$, $T \in V'$ we write $\underbrace{T(f)}_{\in \mathbb{F}} = (f, T)_{V,V'}$ "natural pairing"

Recall from Linear Algebra:

- In finite dimensional vector spaces, elements in V (in some basis, column vector) can be identified with elements in V' (row vector)
- But in infinite dimensional spaces, V' can be "larger" than V (dual to basis in V is not necessarily a basis)

Definition 2.26:

The elements of the dual space $\mathcal{S}'(\mathbb{R}^d)$ of $\mathcal{S}(\mathbb{R}^d)$ are called tempered distributions (or "generalized functions").

Examples:

- Let $(1+|x|^2)^{-m} g(x) \in L^1(\mathbb{R}^d)$ for some $m \in \mathbb{N}$; define

$$T_g : \mathcal{S}(\mathbb{R}^d) \to \mathbb{C},\ f \mapsto \int g(x) f(x)\,dx$$

↳ T_g linear clear

↳ T_g continuous? If $f_n \xrightarrow{n\to\infty} f$ in $\mathcal{S}$, does $T_g(f_n - f) \to 0$ (as a sequence in $\mathbb{C}$)?

$$|T_g(f_n - f)| = \left|\int g(x)\big(f_n(x) - f(x)\big)\,dx\right| \le \int |g(x)|\,|f_n(x) - f(x)|\,dx$$

$$\le \underbrace{\int (1+|x|^2)^{-m} |g(x)|\,dx}_{<\infty}\ \underbrace{\|(1+|x|^2)^m |f_n(x) - f(x)|\|_\infty}_{\xrightarrow{n\to\infty} 0}$$

$$\Rightarrow T_g \in \mathcal{S}'$$

- Delta distribution $\delta : \mathcal{S} \to \mathbb{C},\ f \mapsto \delta(f) = f(0)$

$\Rightarrow \delta \in \mathcal{S}'$ clear $(|f_n(0) - f(0)| \le \|f_n - f\|_\infty)$

A useful notation (in the spirit of previous example) is

$\delta(f) = f(0) = \int \delta(x) f(x)\,dx$, and similarly $\int \delta(x-a) f(x)\,dx = f(a) = \delta_a(f)$, $a \in \mathbb{R}^d$,

but keep in mind that $\delta(x)$ is not a function $\mathbb{R}^d \to \mathbb{C}$!

δ can be approximated by functions, e.g., in the following way:

Let $g \in L^1(\mathbb{R})$ ($d=1$ here), $\int g(x)\,dx = 1$ and $g_n(x) = n\,g(nx)$ (a dilation as in HW2)

s.t. $\int g_n(x)\,dx = \int n\,g(nx)\,dx = \int g(y)\,dy = 1$

$$\Rightarrow \lim_{n\to\infty} T_{g_n}(f) = \lim_{n\to\infty} \int g_n(x) \underbrace{f(x)}_{= f(0) + f(x) - f(0)} dx$$

$$= f(0) + \lim_{n\to\infty} \underbrace{\int n\, g(nx) \big(f(x) - f(0)\big) dx}$$

$$= \int g(y) \underbrace{\Big(f\big(\tfrac{y}{n}\big) - f(0)\Big)}_{\xrightarrow{n\to\infty} 0 \text{ pointwise}} dy \xrightarrow{n\to\infty} 0 \text{ by dominated convergence}$$

$$= f(0) = \delta(f)$$

Next: We have two natural notions of convergence (for $(f_n), f_n \in V$, and $(T_n)_n, T_n \in V'$).

<u>Definition 2.29:</u> Let V be a topological vector space. We define:

a) $(f_n)_n$, $f_n \in V$ $\boxed{\text{converges weakly}}$ to $f \in V$ if $\lim\limits_{n\to\infty} T(f_n) = T(f)$ $\forall\, T \in V'$.

We use the notation: $\underset{n\to\infty}{\text{w-lim}}\, f_n = f$ or $f_n \rightharpoonup f$

b) $(T_n)_n$, $T_n \in V'$ is a $\boxed{\text{weak* convergent}}$ sequence with limit $T \in V'$ if

$\lim\limits_{n\to\infty} T_n(f) = T(f)$ $\forall\, f \in V$

We use the notation: $\underset{n\to\infty}{\text{w*-lim}}\, T_n = T$ or $T_n \xrightarrow{*} T$

<u>Ex.:</u> $T_{g_n} \xrightarrow{*} \delta$

Next: extend $\mathcal{F}$ and ∂_x^α to operators $\mathcal{S}' \to \mathcal{S}'$

Theorem 2.30:

Let $A: S \to S$ be linear and continuous. Then the $\boxed{\text{adjoint } A': S' \to S'}$, defined via

$$\overbrace{\underbrace{(A' \underbrace{T}_{\in S'})}_{\in S'}(\underbrace{f}_{\in S})}^{\in \mathbb{C}} := \underbrace{T(Af)}_{= (f, A'T)_{S,S'} = (Af, T)_{S,S'}} \quad \forall f \in S,$$

is a weak* continuous linear map.

Proof: First, $A'T \in S'$, since $T \circ A$ composition of continuous maps.

Sequential continuity: Let $T_n \xrightarrow{*} T$, then $\forall f \in S$:

$$(A'T_n)(f) := T_n(Af) \xrightarrow{n \to \infty} T(Af) = (A'T)(f), \text{ so } A'T_n \xrightarrow{*} A'T \checkmark$$

Problem: topology in S' not given by a metric, so sequential continuity does not necessarily imply continuity.

But here it does, using the topological concept of nets (proof omitted). $\square$