Foundations of Mathematical Physics, Constructor University Bremen — Session 12

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Next: define $\mathcal{F}, \partial_x^\alpha, M_f : S' \to S'$

We start with $\mathcal{F}$:

Definition 2.31: $\mathcal{F}_{S'} := \mathcal{F}_S'$, meaning for $T \in S'$, we define its Fourier transform $\hat{T} \in S'$ by $\hat{T}(f) = T(\hat{f})$ $\forall f \in S$.

Corollary 2.32: $\mathcal{F}' : S' \to S'$ is a weak*-continuous bijection, and $\underbrace{\hat{T_f} = T_{\hat{f}}}$ for all $f \in S$ (or $f \in L^1$) (recall $T_f(g) := \int fg$).

i.e., $\hat{T}_f(g) = T_{\hat{f}}(g) := \int \hat{f} g$ $\forall g \in S$

Proof: $\mathcal{F} : S \to S$ is continuous and linear, so we conclude with Thm. 2.30 that $\mathcal{F}' : S' \to S'$ is weak*-continuous.

Bijective? $\left(\mathcal{F}^{-1\prime} \mathcal{F}' T\right)(f) = (\mathcal{F}' T)(\mathcal{F}^{-1} f) = T(\mathcal{F}\mathcal{F}^{-1} f) = T(f)$

$\Rightarrow$ yes, with continuous inverse $\mathcal{F}'^{-1} = \mathcal{F}^{-1\prime}$.

Also, for $f \in S$ or $f \in L^1$:

$$\hat{T}_f(g) = (\mathcal{F} T_f)(g) = T_f(\mathcal{F} g) = \int f(x)\, \hat{g}(x)\, dx \overset{\text{Plancherel}}{=} \int \hat{f}(x) g(x) = T_{\hat{f}}(g) \quad \forall g \in S \qquad \square$$

Ex.: Fourier transform of δ ($\delta(f) = f(0)$)

$$\Rightarrow \hat{\delta}(f) = \delta(\hat{f}) = \hat{f}(0) = \int \underbrace{(2\pi)^{-\frac{d}{2}}}_{g(x)} f(x)\, dx = T_g(f)$$

$\Rightarrow T_g$ with $g(x) = (2\pi)^{-\frac{d}{2}}$ is the Fourier transform of δ, or "$\hat{\delta}(k) = (2\pi)^{-\frac{d}{2}}$"

Next: derivatives

Note: $\partial_x^\alpha : \mathcal{S} \to \mathcal{S}$ is linear (clear) and continuous, since

$$\|\partial_x^\alpha f\|_{\gamma,\beta} = \|x^\gamma \partial_x^\beta \partial_x^\alpha f\|_\infty = \|f\|_{\gamma,\alpha+\beta}$$ (i.e., continuity on $\mathcal{S}$ follows as usual from sequential continuity)

Definition 2.34: $\tilde{\partial}_x^\alpha := \left((-1)^{|\alpha|} \partial_x^\alpha\right)' : \mathcal{S}' \to \mathcal{S}'$, i.e., for $T \in \mathcal{S}'$ the distributional derivative $\tilde{\partial}_x^\alpha T$ is defined by $\left(\tilde{\partial}_x^\alpha T\right)(f) := T\left((-1)^{|\alpha|} \partial_x^\alpha f\right)$ $\forall f \in \mathcal{S}$.

Corollary 2.35: $\tilde{\partial}_x^\alpha : \mathcal{S}' \to \mathcal{S}'$ is weak*-continuous and $\tilde{\partial}_x^\alpha T_g = T_{\partial_x^\alpha g}$ $\forall g \in \mathcal{S}$.

Proof: Weak*-continuity follows again from Thm. 2.30.

Also, $$\left(\tilde{\partial}_x^\alpha T_g\right)(f) = T_g\left((-1)^{|\alpha|} \partial_x^\alpha f\right) = \int g(x) (-1)^{|\alpha|} \partial_x^\alpha f(x)\, dx$$

$$\overset{|\alpha| \text{ times integration by parts}}{=} \int \left(\partial_x^\alpha g(x)\right) f(x)\, dx = T_{\partial_x^\alpha g}(f) \quad \forall f \in \mathcal{S}.$$

Ex.:
- For $\Theta(x) = \mathbb{1}_{[0,\infty)}(x) := \begin{cases} 1 & \text{for } x \geq 0 \\ 0 & \text{for } x < 0 \end{cases}$, we find $\frac{d}{dx}\Theta = \delta$, see HW.
- $\tilde{\partial}_x^\alpha \delta$? See HW.

Summary: we have defined $\tilde{\mathcal{F}} = \mathcal{F}' : \mathcal{S}' \to \mathcal{S}'$ and $\tilde{\partial}_x^\alpha = (-1)^{|\alpha|} \partial_x^{\alpha\,\prime} : \mathcal{S}' \to \mathcal{S}'$.

Furthermore one can show:

- Fixing $h \in S$, we can define the convolution $h \tilde{*} \cdot : S' \to S'$ via $(h \tilde{*} T)(f) = T(\tilde{h} * f)$ with $\tilde{h}(x) = h(-x)$. This definition is chosen such that $\underbrace{h \tilde{*} T_g = T_{g*h}}$ for $g \in S$.

  $(h *' T_g)(f) := T_g(\tilde{h} * f) := \int dx\, g(x) \int dy\, h(y-x) f(y)$
  $= \int dy\, f(y) \int dx\, h(y-x) g(x)$

- Fixing $g \in C^\infty_{pol}$, we define $\tilde{M}_g = M_g'$, i.e., $(M_g T)(f) = T(\underbrace{M_g f}_{= gf})$.

  Note: gT well-defined for $g \in C^\infty_{pol}$, but product of distributions a-priori undefined (much research effort to define it at least for some distributions, e.g., Hairer's regularity structures).

Both are weak* continuous maps.

Note: $\{T_f \in S' : f \in S\}$ is dense in S' w.r.t. weak*-topology (not obvious, proof omitted).
Thus, T_f allows us to identify S with some subset of S'.
Because of density and continuity of the adjoint, the definition $A' T_f = T_{Af}$ uniquely defines A' on all of S'. $\leftarrow$ This is why we defined, e.g., $\tilde{\partial}_x^\alpha = (-1)^{|\alpha|} \partial_x^{\alpha\,\prime}$.

From now on, we will forget about $\sim$ or $'$ in the notation for the adjoint.

$\Rightarrow$ We have defined $\mathcal{F}T$, $\partial_x^\alpha T$, $h * T$ for $h \in S$, gT for $g \in C^\infty_{pol}$ $(T \in S')$.

With that we can solve the free Schrödinger equation on S':

Theorem 2.40:

Let $\psi_0 \in S'$, then the unique global solution to the free Schrödinger equation $i\partial_t \psi = -\frac{1}{2}\Delta\psi$ (in the sense of distributions) with $\psi(0)=\psi_0$ is $\psi(t) = \mathcal{F}^{-1} e^{-i\frac{k^2}{2}t}\mathcal{F}\psi_0$, with $\psi \in C^\infty(\mathbb{R}_t, S'(\mathbb{R}^d))$.

Proof: First, note that $\psi(t) = \mathcal{F}^{-1}e^{-i\frac{k^2}{2}t}\mathcal{F}\psi_0 \in S'$ since $\mathcal{F}, \mathcal{F}^{-1}, M_f : S' \to S'$.

Next, let us check if this $\psi(t)$ solves the SE. For any $f \in S$, we find

$$i\frac{d}{dt}(f, \psi(t))_{S,S'} = i\frac{d}{dt}\left(f, \mathcal{F}^{-1}e^{-i\frac{k^2}{2}t}\mathcal{F}\psi_0\right)_{S,S'}$$

by def.
$$= i\frac{d}{dt}\left(\mathcal{F}e^{-i\frac{k^2}{2}t}\mathcal{F}^{-1}f, \psi_0\right)_{S,S'}$$

continuity of $\psi_0 : S \to \mathbb{C}$
$$= \left(\mathcal{F}\left(i\frac{d}{dt}e^{-i\frac{k^2}{2}t}\right)\mathcal{F}^{-1}f, \psi_0\right)_{S,S'}$$

$$= \left(\mathcal{F}e^{-i\frac{k^2}{2}t}\underbrace{\frac{k^2}{2}\mathcal{F}^{-1}f}_{=\mathcal{F}^{-1}(-\frac{\Delta}{2}f)}, \psi_0\right)_{S,S'}$$

$$= \left(-\frac{\Delta}{2}f, \mathcal{F}^{-1}e^{-i\frac{k^2}{2}}\mathcal{F}\psi_0\right)_{S,S'}$$

by def. of the distributional derivative
$$= \left(f, -\frac{\Delta}{2}\psi(t)\right)_{S,S'}.$$

Similarly $\left(i\frac{d}{dt}\right)^k (f, \psi(t))_{S,S'} = \left(\left(-\frac{\Delta}{2}\right)^k f, \psi(t)\right)_{S,S'}$, so $\psi(t) \in C^\infty(\mathbb{R}_t, S'(\mathbb{R}^d))$. $\square$