Next, we establish a more direct connection to velocity:

Consider the probability density $\rho_\psi(t,x) = |\psi(t,x)|^2$ and $i\partial_t \psi(t,x) = \left(-\frac{\Delta}{2} + V\right)\psi(t,x)$, $V: \mathbb{R}^d \to \mathbb{R}$

$$\Rightarrow \partial_t \rho_\psi(t,x) = \partial_t |\psi(t,x)|^2 = \overline{\left(\partial_t \psi(t,x)\right)}\, \psi(t,x) + \overline{\psi(t,x)}\left(\partial_t \psi(t,x)\right)$$

$$= \frac{i}{2}\overline{\left(-\Delta\psi(t,x) + V(x)\psi(t,x)\right)}\, \psi(t,x) - \frac{i}{2}\, \overline{\psi(t,x)}\left(-\Delta\psi(t,x) + V(x)\psi(t,x)\right)$$

$\frac{i}{2}\bar{z} - \frac{i}{2} z = \operatorname{Im} z$

$$= \operatorname{Im} \overline{\psi(t,x)}\left(-\Delta\psi(t,x)\right)$$

$$= -\nabla \underbrace{\operatorname{Im} \overline{\psi(t,x)}\left(\nabla\psi(t,x)\right)}_{=: j_\psi(t,x) = \text{current}} \qquad \left(\text{since } \overline{\nabla\psi}\, \nabla\psi \in \mathbb{R}\right)$$

$$\Rightarrow \boxed{\partial_t \rho_\psi + \nabla j_\psi = 0 \text{, continuity equation}}$$

Note: The continuity eq. implies:

$$\underbrace{\partial_t \int_\Lambda \rho_\psi\, dx}_{\text{change of mass/probability/... in } \Lambda \subset \mathbb{R}^d \text{ (compact)}} = -\int_\Lambda \nabla j_\psi\, dx \overset{\text{Gauss (Stokes)}}{=} \underbrace{-\int_{\partial\Lambda} j_\psi\, d\sigma}_{\text{flow through boundary of } \Lambda}$$

change of mass/probability/... in $\Lambda \subset \mathbb{R}^d$ (compact) = flow through boundary of Λ

Now: current = density · velocity, i.e., $j_\psi = \rho_\psi \cdot v_\psi$

$$\Rightarrow \text{velocity vector field } v_\psi(t,x) = \frac{j_\psi(t,x)}{\rho_\psi(t,x)} = \frac{\operatorname{Im} \overline{\psi(t,x)}\, \nabla\psi(t,x)}{\overline{\psi(t,x)}\, \psi(t,x)} = \operatorname{Im} \underbrace{\frac{\nabla\psi(t,x)}{\psi(t,x)}}$$

looks dangerous at zeros of ψ, but since $\rho_\psi = 0$ at zeros of ψ, the velocity field never needs to be evaluated at the zeros.

Let us approximate v_ψ for large t.

$$v_\psi(t,x) = \operatorname{Im} \frac{D\psi(t,x)}{\psi(t,x)}$$

We skip the rigorous estimate

$$\approx \operatorname{Im} \frac{D_x\left((it)^{-\frac{d}{2}} e^{i\frac{x^2}{2t}} \hat{\psi}_0\left(\frac{x}{t}\right)\right)}{(it)^{-\frac{d}{2}} e^{i\frac{x^2}{2t}} \hat{\psi}_0\left(\frac{x}{t}\right)}$$

$$= \operatorname{Im} \frac{i\frac{x}{t} e^{i\frac{x^2}{2t}} \hat{\psi}_0\left(\frac{x}{t}\right) + e^{i\frac{x^2}{2t}} D_x \psi_0\left(\frac{x}{t}\right)}{e^{i\frac{x^2}{2t}} \hat{\psi}_0\left(\frac{x}{t}\right)}$$

$$\approx \frac{x}{t} + \mathcal{O}\left(\frac{1}{t}\right)$$

↳ So for example, in this sense classical trajectories appear in QM

3. The Schrödinger Equation with Potential

Next, we want to understand the Schrödinger equation

$$i\partial_t \psi(t,x) = -\frac{\Delta}{2}\psi(t,x) + V(x)\psi(t,x) = H\psi(t,x) \qquad (\text{here: } V \text{ time-independent})$$

Note: • Fourier transformation turns V into convolution $\Rightarrow$ not easy to find solutions for $V \not\equiv 0$.

• Since $|\psi(t,x)|^2$ is a probability density, we want to understand the SE on L^2.
(For the free SE we had $\|\psi(t)\|_{L^2} = \|\psi(0)\|_{L^2}$ if $\psi(0) \in L^2$, and that should also hold for $V \not\equiv 0$.)

Ideas that we will develop in this chapter:

- As we have done for the free SE, we try to make sense of e^{-iHt} for a large class of V, s.t. we can define $\psi(t) = e^{-iHt}\psi(0)$.
- We regard L^2 as a subspace of S', s.t. the SE holds in the sense of distributions; but hopefully it also holds on L^2, at least for some initial data.

First, we want to embed the free SE in the L^2 framework, so we discuss Hilbert spaces (and operators on them) in general, and then how to define $\mathcal{F}: L^2 \to L^2$.

3.1 Hilbert and Banach Spaces

Recall: • $\boxed{\text{Banach space}}$ = $\underbrace{\text{complete}}$ normed vector space
every Cauchy sequence converges

• $\boxed{\text{Hilbert space}}$ = Banach space with scalar product $\langle \cdot , \cdot \rangle : \mathcal{H} \times \mathcal{H} \to \mathbb{C}$ s.t. $\|\psi\| = \sqrt{\langle \psi, \psi \rangle}$.
or any other field

Convention: $\langle \lambda \psi, \varphi \rangle = \bar{\lambda} \langle \psi, \varphi \rangle$. "antilinearity in the first argument"

Examples: • $\mathbb{C}^n$ with $\langle x, y \rangle_{\mathbb{C}^n} = \sum_{j=1}^{n} \overline{x_j} y_j$

• ℓ^2 with $\langle x, y \rangle_{\ell^2} = \sum_{j=1}^{\infty} \overline{x_j} y_j$

• $L^2(M, \mu)$ for some measure space (M, μ), with $\langle \psi, \varphi \rangle_{L^2} = \int_M \overline{\psi(x)} \varphi(x) d\mu$

Let us first prove some standard properties.
In the following, let $\mathcal{H}$ be a Hilbert space.

$$\boxed{\text{Definition 3.3: A sequence } (\varphi_j)_j \text{ in } \mathcal{H} \text{ is called orthonormal sequence (ONS) if } \langle \varphi_i, \varphi_j \rangle = \delta_{ij} \quad \forall i, j.}$$

The following properties hold:

• orthonormal decomposition: $\forall \psi \in \mathcal{H}$: $\psi = \underbrace{\sum_{j=1}^{n} \langle \varphi_j, \psi \rangle \varphi_j}_{=: \psi_n} + \underbrace{\left(\psi - \sum_{j=1}^{n} \langle \varphi_j, \psi \rangle \varphi_j \right)}_{=: \psi_n^\perp}$

with $\langle \psi_n, \psi_n^\perp \rangle = \langle \psi_n, \psi - \psi_n \rangle = \langle \psi_n, \psi \rangle - \langle \psi_n, \psi_n \rangle = \sum_{j=1}^{n} \overline{\langle \varphi_j, \psi \rangle} \langle \varphi_j, \psi \rangle - \sum_{i,j=1}^{n} \overline{\langle \varphi_j, \psi \rangle} \underbrace{\langle \varphi_j, \varphi_i \rangle}_{= \delta_{ij}} \langle \varphi_i, \psi \rangle$
$= 0$

$\Rightarrow \langle \psi, \psi \rangle = \langle \psi_n + \psi_n^\perp, \psi_n + \psi_n^\perp \rangle = \langle \psi_n, \psi_n \rangle + \langle \psi_n^\perp, \psi_n^\perp \rangle$