Let $\mathcal{H}$ be a Hilbert space.

Recall: For a subset $M \subset \mathcal{H}$, we def. $M^\perp := \{\psi \in \mathcal{H} : \langle \varphi, \psi \rangle = 0 \ \forall \varphi \in M\}$ ← $M^\perp$ is a closed subspace

Theorem 3.15: Let $M \subset \mathcal{H}$ be a closed subspace. Then $\mathcal{H} = M \oplus M^\perp$, meaning $\forall \psi \in \mathcal{H}$

we have $\psi = \varphi + \varphi^\perp$ with unique $\varphi \in M$, $\varphi^\perp \in M^\perp$.

Proof: We give the proof only for separable $\mathcal{H}$.

$\Rightarrow$ also $M, M^\perp$ are separable Hilbert spaces with ONBs $(\varphi_i)_i$, $(\varphi_j^\perp)_j$.

Choose any $\psi \in \mathcal{H}$, def. $\varphi = \sum_{i=1}^\infty \langle \varphi_i, \psi \rangle \varphi_i$, $\varphi^\perp = \sum_{j=1}^\infty \langle \varphi_j^\perp, \psi \rangle \varphi_j^\perp$, then

$\psi = \varphi + \varphi^\perp$ if $(\varphi_i)_i \cup (\varphi_j^\perp)_j$ is an ONB of $\mathcal{H}$.

Check with Proposition 3.10: Let $\underbrace{\langle \varphi_i, \psi \rangle = 0}$ $= \langle \varphi_j^\perp, \psi \rangle \ \forall i, j$

$\Rightarrow \langle \varphi, \psi \rangle = 0 \ \forall \varphi \in M \Rightarrow \psi \in M^\perp$

but since also $\langle \varphi^\perp, \psi \rangle = 0 \ \forall \varphi^\perp \in M^\perp \Rightarrow \psi = 0$

Uniqueness: suppose there is another decomposition $\psi = \underbrace{\tilde{\varphi}}_{\in M} + \underbrace{\tilde{\varphi}^\perp}_{\in M^\perp}$

$\Rightarrow \varphi + \varphi^\perp = \tilde{\varphi} + \tilde{\varphi}^\perp$, i.e., $M \ni \varphi - \tilde{\varphi} = \tilde{\varphi}^\perp - \varphi^\perp \in M^\perp$

but $M \cap M^\perp = \{0\}$ so $\varphi = \tilde{\varphi}$ and $\varphi^\perp = \tilde{\varphi}^\perp$. $\square$

Next topic: operators between normed spaces

In ∞-dimensions, there are not just bounded, but also unbounded operators.

Bounded ones are nicer, so let us consider those first.

(Later: Hamiltonians H will be unbounded, but e^{-iHt} will be bounded.)

<u>Definition 3.16:</u> Let X and Y be normed spaces. A linear operator $L: X \to Y$ is bounded if $\exists C < \infty$ with $\underbrace{\|Lx\|_Y}_{\text{norm on } Y} \leq C \underbrace{\|x\|_X}_{\text{norm on } X}$ $\forall x \in X$.

Example: Multiplication operators $M_V: L^p \to L^p$ for $V \in L^\infty$ are bounded (HW 4 Problem 3).

<u>Proposition 3.17:</u> The space $\mathcal{B}(X,Y) = \{L: X \to Y, L \text{ linear and bounded}\}$ with norm $\boxed{\|L\|_{\mathcal{B}(X,Y)} := \sup_{\substack{x \in X \\ \|x\|_X = 1}} \|Lx\|_Y}$ is itself a normed space.

If $\underbrace{Y \text{ is a Banach space}}_{\text{not necessarily } X}$, also $\mathcal{B}(X,Y)$ is a Banach space.

Proof: HW 7

Why are bounded operators so interesting? Because these are also the continuous ones!

(And since we deal with linear ones, it is enough to check continuity at 0.)

Lemma 3.18: Let $L: X \to Y$ be linear (X, Y normed spaces). Then the following statements are equivalent: (i) L is continuous at 0.

(ii) L is continuous.

(iii) L is bounded.

Proof: (iii) $\Rightarrow$ (i): Let $\|x_n\|_X \to 0 \Rightarrow \|Lx_n\|_Y \underset{\text{bounded}}{\leq} \|L\|_\delta \|x_n\|_X \to 0$

(i) $\Rightarrow$ (ii): Let $\|x_n - x\|_X \to 0 \Rightarrow \|Lx_n - Lx\|_Y \overset{\text{linearity}}{=} \|L(x_n - x)\|_Y \xrightarrow{\text{continuity at } 0} 0$

(ii) $\Rightarrow$ (iii): suppose L not bounded, then $\exists$ a sequence $(x_n)_n$ with $\|x_n\|_X = 1 \ \forall n \in \mathbb{N}$ and $\underbrace{\|Lx_n\|_Y \geq c(n)}$ for some $c(n) \xrightarrow{n \to \infty} \infty$. Defining $z_n := \frac{x_n}{\|Lx_n\|_Y}$, we have $\|z_n\| = \frac{\|x_n\|_X}{\|Lx_n\|_Y} \leq \frac{1}{c(n)}$, i.e., $z_n \xrightarrow{n \to \infty} 0$. But $\|Lz_n\|_Y = \frac{\|Lx_n\|_Y}{\|Lx_n\|_Y} = 1$, which contradicts continuity (at 0). $\square$

Note: by using subsequences (rescaling the index) we could even use $c(n) = n$.

What do unbounded operators look like? Much more later, here just one example:

Define $\ell_0 = \{ (x_n)_n \in \ell^1 : \exists N \in \mathbb{N} \text{ s.t. } x_n = 0 \ \forall n \geq N \}$ with the norm

$\|(x_n)_n\|_{\ell^1} = \overbrace{\sum_{n=1}^{\infty} |x_n|}^{\text{actually just a finite sum}}$. Define $T: \ell_0 \to \ell_0$, $x \mapsto Tx = (x_1, 2x_2, 3x_3, \dots)$.

But if $(e^{(n)}_k)_k$ is the sequence with $e^{(n)}_k = \begin{cases} 1 & \text{for } k = n \\ 0 & \text{otherwise} \end{cases}$, in particular $\|e^{(n)}\| = 1$,

then $\|Te^{(n)}\| = n$, i.e., T is unbounded.

In the last chapter, we defined operators on S' by defining them on a dense subset and extending them by continuity (but we did not fully prove this). This can also be done here (for bounded = continuous operators):

Theorem 3.20: Let Z be a dense subspace of a normed space X, and let Y be a Banach space. Let $L: Z \to Y$ be a linear bounded operator. Then L has a unique linear bounded extension $\tilde{L}: X \to Y$ with $\underbrace{\tilde{L}|_Z = L}_{\tilde{L} \text{ and } L \text{ coincide on } Z}$ and $\|\tilde{L}\|_{\mathcal{L}(X,Y)} = \|L\|_{\mathcal{L}(Z,Y)}$.

Proof: Idea: using continuity we "fill in the gaps."

Choose some $x \in X$, then $\exists$ sequence $(z_n)_n$ in Z with $\|z_n - x\|_X \to 0$ (using just density of Z in X; note: $x \in X$ is fixed, no completeness necessary).

$\Rightarrow (z_n)_n$ converges $\Rightarrow (z_n)_n$ is a Cauchy sequence.

$\Rightarrow \|Lz_n - Lz_m\|_Y \overset{\text{linearity}}{=} \|L(z_n - z_m)\|_Y \leq \|L\|_{\mathcal{L}(Z,Y)} \|z_n - z_m\|_Z$, i.e., also $(Lz_n)_n$ is a Cauchy sequence in Y. Since Y is complete, $Lz_n \to y \in Y$.

But is this y independent of the choice of sequence?

Yes: if $\|z_n' - x\|_X \to 0$, also the sequence $(z_1, z_1', z_2, z_2', z_3, z_3', \ldots)$ converges to x and as above $(Lz_1, Lz_1', Lz_2, Lz_2', \ldots)$ converges to some $\tilde{y} \in Y$. But every subsequence of a convergent sequence converges to the same limit.

So we def. $\hat{L}x := y$ with this construction.

$\|\hat{L}\|_{\mathcal{L}(X,Y)} \leq \|L\|_{\mathcal{L}(Z,Y)}$

(and $\|L\|_{\mathcal{L}(Z,Y)} \leq \|\hat{L}\|_{\mathcal{L}(X,Y)}$ clear by def.)

↳ linearity clear

↳ boundedness: $\|\hat{L}x\|_Y = \lim_{n\to\infty} \underbrace{\|Lz_n\|_Y}_{\leq \|L\|_{\mathcal{L}(Z,Y)} \|z_n\|_Z} \leq \|L\|_{\mathcal{L}(Z,Y)} \|x\|_X \Rightarrow \hat{L}$ continuous

and continuity on a dense subset implies that this is the unique extension. $\square$