Last time we proved:

Let Z be a dense subspace of a normed space X, and let Y be a Banach space.
Let $L: Z \to Y$ be a linear bounded operator. Then L has a unique linear bounded extension $\tilde{L}: X \to Y$ with $\tilde{L}|_Z = L$ and $\|\tilde{L}\|_{\mathcal{L}(X,Y)} = \|L\|_{\mathcal{L}(Z,Y)}$.

Now, e.g., extension of the Fourier transform from $\mathcal{S}$ to L^2 follows as a simple corollary.
Let us first note:

Theorem 3.21: $\underbrace{C_c^\infty(\mathbb{R}^d)}_{\text{smooth functions with compact support}}$ is dense in $L^p(\mathbb{R}^d)$, $1 \le p < \infty$.

Proof: From HW 4, Problem 4(b), we know that C_c^∞ is dense in C_c $\underbrace{\text{w.r.t. } \|\cdot\|_{L^p}}$.

density is defined w.r.t. a norm, or generally a topology (a subset might be dense w.r.t. one norm, but not another)

(We used convolution there to "smoothen out" (or "mollify") $f \in L^p$.)

It is also a standard result that C_c is dense in L^p, which implies that C_c^∞ is dense in L^p (by a triangle argument). $\square$

Then we have

Theorem 3.22: The Fourier transform $\mathcal{F}: (\mathcal{S}(\mathbb{R}^d), \|\cdot\|_{L^2(\mathbb{R}^d)}) \longrightarrow L^2(\mathbb{R}^d)$ can be uniquely extended to a bounded linear operator $L^2 \to L^2$.

Furthermore:
- $\|\mathcal{F}f\|_{L^2} = \|f\|_{L^2} \quad \forall f \in L^2$
- $\mathcal{F}\mathcal{F}^{-1} = \mathcal{F}^{-1}\mathcal{F} = \mathrm{id}_{L^2}$
- $(\mathcal{F}f)(k) = \underbrace{\lim_{N\to\infty}}_{L^2 \text{ limit, not pointwise}} (2\pi)^{-\frac{d}{2}} \int_{|x|\le N} e^{-ikx} f(x)\,dx \quad \forall f \in L^2.$

Proof: $C_c^\infty \subset S \subset L^2$, so with Thm. 3.21 also S is dense in L^2 and we can apply Thm. 3.20. (Note: $\mathcal{F}: (S, \|\cdot\|_{L^2}) \to L^2$ is indeed bounded, since $\|\mathcal{F}f\|_{L^2} = \|f\|_{L^2}$ (Plancherel).)
Also: $\mathcal{F}\mathcal{F}^{-1}|_S = \mathcal{F}^{-1}\mathcal{F}|_S = \mathrm{id}_{L^2}|_S$, but since $\mathcal{F}, \mathcal{F}^{-1}, \mathrm{id}$ continuous, equality holds on L^2.

Limit formula follows directly from $\|\mathcal{F}f\|_{L^2} = \|f\|_{L^2}$: let us denote $f_N(x) = f(x)\underbrace{\mathbb{1}_{B_N(0)}(x)}_{=\begin{cases}1 & \text{for } |x|\le N\\ 0 & \text{else}\end{cases}}$. Then $\lim_{N\to\infty} \|\mathcal{F}f - \mathcal{F}f_N\|_{L^2} = \lim_{N\to\infty} \|f - f_N\|_{L^2} = 0$. $\square$

Note: • one can of course use any other suitable limit formula for explicit computations.
• so even for functions $\notin L^1$, we have defined $\int f(x) e^{-ikx}\,dx$.

Note that $\mathcal{F}: L^2 \to L^2$ is a unitary operator:

Definition 3.23: Let $\mathcal{H}_1, \mathcal{H}_2$ be Hilbert spaces. A linear bounded operator $U \in \mathcal{B}(\mathcal{H}_1, \mathcal{H}_2)$ is called **unitary** if it is surjective and isometric (isometric meaning $\|U\psi\|_{\mathcal{H}_2} = \|\psi\|_{\mathcal{H}_1}$ $\forall \psi \in \mathcal{H}_1$).

Note: • injective follows from $\|U\psi\|_{\mathcal{H}_2} = \|\psi\|_{\mathcal{H}_1}$, so unitary operators are bijective
• with the polarization identity isometry $\Leftrightarrow$ preservation of inner product:

$$\langle U\psi, U\varphi\rangle_{\mathcal{H}_2} = \langle \psi, \varphi\rangle_{\mathcal{H}_1} \quad \forall \psi, \varphi \in \mathcal{H}_1$$

Having $\mathcal{F}: L^2 \to L^2$, we can now solve the free Schrödinger equation on L^2:

For any $t \in \mathbb{R}$, the free propagator on L^2 is $P_f(t): L^2(\mathbb{R}^d) \to L^2(\mathbb{R}^d)$, $P_f(t) = \mathcal{F}^{-1} e^{-i\frac{k^2}{2}t}\mathcal{F}$.

$\Rightarrow P_f(t)$ is clearly unitary ($|e^{-i\frac{k^2}{2}t}| = 1$ and $\mathcal{F}$ isometric) for any $t \in \mathbb{R}$.

To talk about continuity and differentiability of $P_f(t)$, i.e., of $P_f: \mathbb{R} \to \mathcal{B}(L^2)$, we need to distinguish different notions of convergence for bounded operators.

Definition 3.26: Let $(A_n)_n$ be a sequence in $\mathcal{L}(\mathcal{H})$ and $A \in \mathcal{L}(\mathcal{H})$.

a) $(A_n)_n$ converges in norm (or "uniformly") to A if $\lim_{n\to\infty} \|A_n - A\|_{\mathcal{L}(\mathcal{H})} = 0$.

Notation: $\lim_{n\to\infty} A_n = A$, or $A_n \longrightarrow A$.

b) $(A_n)_n$ converges strongly (or "pointwise") to A if $\lim_{n\to\infty} \|A_n\psi - A\psi\|_{\mathcal{H}} = 0 \ \forall \psi \in \mathcal{H}$.

Notation: $\underset{n\to\infty}{\text{s-lim}}\, A_n = A$, or $A_n \xrightarrow{s} A$.

c) $(A_n)_n$ converges weakly to A if $\lim_{n\to\infty} |\langle \varphi, (A_n - A)\psi\rangle| = 0 \quad \forall \varphi, \psi \in \mathcal{H}$.

Notation: $\underset{n\to\infty}{\text{w-lim}}\, A_n = A$, or $A_n \xrightarrow{w} A$.

Note: • $|\langle\varphi, (A_n - A)\psi\rangle| \le \|\varphi\| \, \|(A_n - A)\psi\|_{\mathcal{H}} \le \|\varphi\| \, \|\psi\| \, \|A_n - A\|_{\mathcal{L}(\mathcal{H})}$,

so norm convergence $\Rightarrow$ strong convergence $\Rightarrow$ weak convergence.

But the other way around is not true; come up with counterexamples in HW8.

Let us now check continuity and differentiability of $P_f : \mathbb{R} \to \mathcal{L}(L^2)$:

• Uniformly continuous? $$\|P_f(t+h) - P_f(t)\|_{\mathcal{L}(L^2)} = \sup_{\substack{\varphi \in L^2 \\ \|\varphi\|=1}} \|P_f(t+h)\varphi - P_f(t)\varphi\|_{L^2}$$

$$= \sup_{\substack{\varphi \in L^2 \\ \|\varphi\|=1}} \left\| \left(e^{-i\frac{k^2}{2}(t+h)} - e^{-i\frac{k^2}{2}t}\right) \mathcal{F}\varphi \right\|_{L^2}$$

$$= \sup_{\substack{\tilde\varphi \in L^2 \\ \|\tilde\varphi\|=1}} \left\| \left(e^{-i\frac{k^2}{2}(t+h)} - e^{-i\frac{k^2}{2}t}\right) \tilde\varphi \right\|_{L^2}$$

Problem 3 HW4: $\|M_v\|_{\mathcal{L}(L^2)} = \|v\|_\infty$

$$= \sup_{k\in\mathbb{R}^d} \underbrace{\left| e^{-i\frac{k^2}{2}(t+h)} - e^{-i\frac{k^2}{2}t} \right|}_{= |e^{-i\frac{k^2}{2}h}-1|}$$

$$= 2 \quad \text{for all } h\neq 0.$$

So $\lim_{h\to 0} \|P_f(t+h) - P_f(t)\|_{\mathcal{L}(L^2)} = 2$, i.e., $P_f(t)$ is not uniformly continuous.

- Strongly continuous? $\|P_f(t+h)\psi_0 - P_f(t)\psi_0\|_{L^2}^2 = \|\psi(t+h)-\psi(t)\|_{L^2}^2$

$$= \|\mathcal{F}^{-1}\left(e^{-i\frac{k^2}{2}t}e^{-i\frac{k^2}{2}h} - e^{-i\frac{k^2}{2}t}\right)\overbrace{\mathcal{F}\psi_0}^{\hat{\psi}_0}\|^2$$

$$= \int \underbrace{\left|e^{-i\frac{k^2}{2}h}-1\right|^2}_{\xrightarrow{h\to 0} 0} |\hat{\psi}_0(k)|^2 \, dk \xrightarrow{h\to 0} 0,$$

by dominated convergence ($\hat{\psi}\in L^2 \Leftrightarrow \psi\in L^2$)

i.e., $P_f(t)$ is strongly continuous on L^2 $\Leftrightarrow \psi(t)$ is continuous $\forall\, \psi_0\in L^2$.

- Strongly differentiable? $\left(\frac{\|P_f(t+h)\psi_0 - P_f(t)\psi_0\|_{L^2}}{h}\right)^2 = \left(\frac{\|\psi(t+h)-\psi(t)\|_{L^2}}{h}\right)^2$

$$= \int \underbrace{\left|\frac{e^{-i\frac{k^2}{2}h}-1}{h}\right|^2}_{\xrightarrow{h\to 0} \frac{k^4}{4}} |\hat{\psi}_0(k)|^2 \, dk,$$

but dominated convergence only applies if $k^4|\hat{\psi}_0(k)|^2$ is integrable, i.e., $k^2\hat{\psi}_0(k)\in L^2$.

$\Rightarrow \psi(t)$ is differentiable only for $\psi_0 \in H^2 := \{\psi\in L^2 : k^2\hat{\psi}_0(k)\in L^2\}$.

And for $\psi_0\in H^2$ we have

$$-\tfrac{1}{2}\underbrace{\Delta}_{\text{distributional derivative}}\psi(t) = -\tfrac{1}{2}\Delta\,\mathcal{F}^{-1}\underbrace{e^{-i\frac{k^2}{2}t}\mathcal{F}\psi_0}_{\hat{\psi}(t)} = \overbrace{\mathcal{F}^{-1}\frac{k^2}{2}\underbrace{e^{-i\frac{k^2}{2}t}\hat{\psi}_0}_{\hat{\psi}(t)}}^{\in L^2} = \overbrace{i\frac{d}{dt}\psi(t)}^{\in L^2}.$$

$\Rightarrow$ The free SE holds as equality of L^2 vectors.

Conclusion: For $\psi_0 \in H^2$, $\psi(t)$ solves the free Schrödinger equation $\forall t$ in the L^2 sense.

If $L^2 \ni \psi_0 \notin H^2$, then $\psi(t)$ solves the free Schrödinger equation in the sense of distributions only (as noted before).