Recall: Let $H: D(H) \to \mathcal{H}$ with $D(H) \subset \mathcal{H}$ dense be linear. H generates a strongly continuous group of unitaries $U(t)$ if

i) $D(H) = \{ \psi \in \mathcal{H} : t \mapsto U(t)\psi \text{ is differentiable} \}$,

ii) For $\psi \in D(H)$, we have $i \frac{d}{dt} U(t)\psi = U(t) H \psi$.

Let us collect some important properties of generators:

Proposition 3.33: Let H be generator of $U(t)$. Then

i) $U(t) D(H) = D(H) \ \forall t$, i.e., $D(H)$ is invariant under $U(t)$,

ii) $[H, U(t)]\psi = 0 \quad \forall \psi \in D(H)$ (where $[A,B] := AB - BA$ is the commutator),

iii) H is symmetric, i.e., $\langle H\psi, \varphi \rangle = \langle \psi, H\varphi \rangle \ \forall \varphi, \psi \in D(H)$,

iv) U is uniquely determined by H, and H is uniquely determined by U.

Proof: HW 9

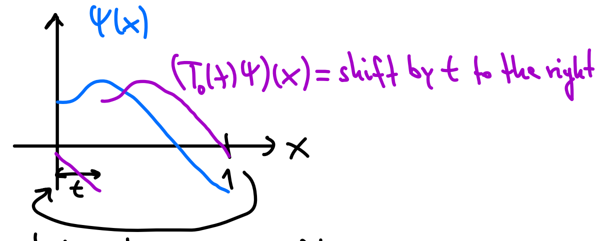
Example: Translation operator on $L^2(\mathbb{R})$

Let us consider $T(t): L^2(\mathbb{R}) \to L^2(\mathbb{R})$, where $(T(t)\psi)(x) := \psi(x-t)$.

We already introduced this operator on $\mathcal{S}$ as the pseudodifferential operator $e^{-it(-i\frac{d}{dx})}$. So we would guess that $D_0 = -i\frac{d}{dx}$ with domain $D(D_0) = H^1(\mathbb{R})$ is the generator of the strongly continuous unitary one-parameter group $T(t)$. This is indeed so; proof in HW 9.

Example: Translation operator on $L^2([0,1])$

We want to define translations as a unitary group:



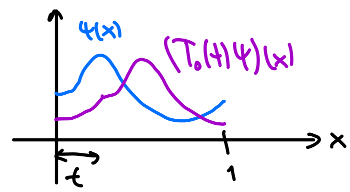
put ψ back in on the left
(otherwise isometry would be violated)

$\Rightarrow$ For $t \in [0,1)$, an obvious translation operator is

$$(T_0(t)\psi)(x) = \begin{cases} \psi(x-t) & \text{if } x-t \in [0,1], \\ \psi(\underbrace{x-t+1}_{\in [0,1) \text{ here}}) & \text{if } x-t<0. \end{cases}$$

How can $-i\frac{d}{dx}$ be a generator here?

$\hookrightarrow$ $-i\frac{d}{dx}(T_0(t)\psi)(x)$ only exists in L^2 if $\psi(0)=\psi(1)$:



More generally, let us define translations for $t \in [0,1)$ as

$$(T_\theta(t)\psi)(x) = \begin{cases} \psi(x-t) & \text{if } x-t \in [0,1] \\ e^{i\theta}\psi(x-t+1) & \text{if } x-t<0 \end{cases}$$, for any phase factor $\theta \in [0,2\pi)$.

$T_\theta(t)$ is clearly unitary, and we define T_θ for all $t \in \mathbb{R}$ by the group property (e.g. if $t,s \in [0,1)$, then T_θ is def. on $[0,2)$ by $T_\theta(t+s) = T_\theta(t)T_\theta(s)$).

Now: $T_\theta \neq T_{\theta'}$ for $\theta \neq \theta'$, so according to Proposition 3.33 iv) their generators must be different.

Consider $D_\theta : \mathcal{D}(D_\theta) \to L^2([0,1])$, $\psi \mapsto -i\frac{d}{dx}\psi$, with domain

$$\mathcal{D}(D_\theta) = \{\psi \in H^1([0,1]) : e^{i\theta}\psi(1) = \psi(0)\}.$$

$H^1([0,1]) := \{\psi \in L^2([0,1]) : \exists \varphi \in H^1(\mathbb{R}) \text{ s.t. } \varphi|_{[0,1]} = \psi\}$

indeed ψ is defined pointwise because of the Sobolev lemma: $H^1(\mathbb{R}) \subset C(\mathbb{R})$.

Then indeed D_θ is the generator of T_θ.

Consistency check: For $\psi, \varphi \in H^1([0,1])$ we find:

$$\langle \psi, -i\frac{d}{dx}\varphi \rangle = \int_0^1 \overline{\psi(x)} \left(-i\frac{d}{dx}\varphi(x)\right) dx$$

integration by parts

$$= -i\left(\overline{\psi(1)}\varphi(1) - \overline{\psi(0)}\varphi(0)\right) + \langle -i\frac{d}{dx}\psi, \varphi \rangle$$

Therefore:

- $-i\frac{d}{dx}$ not symmetric on $D_{max} = H^1([0,1])$ (boundary terms do not vanish), so $-i\frac{d}{dx}$ with domain D_{max} is not a generator
- On D_θ and $D_{min} := \{\psi \in H^1([0,1]) : \psi(0) = 0 = \psi(1)\}$, $-i\frac{d}{dx}$ is symmetric (boundary terms vanish). But on D_{min} it is not a generator, so symmetry is a necessary but not sufficient condition.

  D_{min} is not invariant under any T_θ

Conclusions:

- In applications, we often know operators formally ($-i\frac{d}{dx}$ in this example), but we might not know the domain. It is usually most convenient to choose the domain small (nice regular fct.s), but if we choose it too small (D_{min} in this example), we might not get a generator. Then we try to enlarge the domain, but if we enlarge it too much (D_{max} in this example), we again might not get a generator. Note that enlarging the domain does not necessarily lead to a unique generator (many possibilities D_θ in this example).

- Symmetry is a necessary but not sufficient condition for generators.

  The right class of operators are self-adjoint operators, which we consider next.