## 3.3 Self-adjoint Operators

We consider bounded operators first.

Recall the general definition of the adjoint (here for normed spaces):

**Definition 3.38:** Let $V$ and $W$ be normed spaces and $A \in \mathcal{L}(V,W)$. Then the adjoint operator $A' : W' \to V'$ (where $V'$ and $W'$ are the dual spaces of $V$ and $W$) is defined by

$$A'(w')(v) = w'(Av) \quad \forall v \in V \ \forall w' \in W'.$$

Note:
- For any normed space $V$, the dual space $V'$ is a Banach space (even if $V$ is not). This is so because elements of $V'$ are continuous, i.e., bounded operators $V \to \mathbb{C}$, and $\mathbb{C}$ is complete (cf. Proposition 3.17).
- $A' \in \mathcal{L}(W', V')$ due to the definition
- With the Hahn-Banach theorem one can show that in fact $\|A'\|_{\mathcal{L}(W',V')} = \|A\|_{\mathcal{L}(V,W)}$.

Hilbert spaces are particularly nice because $\mathcal{H}'$ is isometrically isomorphic to $\mathcal{H}$. (We already noted that $L^p \cong (L^q)'$, $\frac{1}{p} + \frac{1}{q} = 1$ in HW 4, so $L^2 \cong (L^2)'$.) So for $A \in \mathcal{L}(\mathcal{H})$, we would like to identify the operator $A' \in \mathcal{L}(\mathcal{H}')$ with an operator on $\mathcal{H}$. Let us first establish this connection; then we can introduce the notion of self-adjointness.

The key theorem is:

**Theorem 3.39: The Riesz Representation Theorem**

Let $\mathcal{H}$ be a Hilbert space and $T \in \mathcal{H}'$. Then there is a unique $\psi_T \in \mathcal{H}$ s.t.

$$T(\varphi) = \langle \psi_T, \varphi \rangle_{\mathcal{H}} \quad \forall \varphi \in \mathcal{H}.$$

## Proof:

First, if $T(\varphi) = 0$ $\forall \varphi \in \mathcal{H}$, then $T = 0$ and $\psi_T = 0$ is the unique vector in the theorem. Otherwise, we want to show that $T$ is the projection on the one-dimensional subspace spanned by some $\psi_T$.

So if we consider the kernel $\mathcal{M} = \ker(T)$ $(:= \{\varphi \in \mathcal{H} : T(\varphi) = 0\})$, a closed subspace of $\mathcal{H}$ (since $T$ is continuous), we need to show that $\mathcal{M}^\perp$ is one-dimensional. If $\mathcal{M} = \mathcal{H}$, i.e., $\dim \mathcal{M}^\perp = 0$, then $\psi_T = 0$, so let us assume $\dim \mathcal{M}^\perp > 0$.

But this follows directly from linearity: Let $\psi, \tilde{\psi} \in \mathcal{M}^\perp \setminus \{0\}$. Then for $\alpha \in \mathbb{C}$,

$$T(\psi - \alpha\tilde{\psi}) = T(\psi) - \alpha T(\tilde{\psi}), \text{ so for } \alpha = \frac{T(\psi)}{T(\tilde{\psi})}, \text{ we have } T(\psi - \alpha\tilde{\psi}) = 0, \text{ i.e.,}$$

$$\psi - \alpha\tilde{\psi} \in \mathcal{M}, \text{ so } \psi - \alpha\tilde{\psi} \in \mathcal{M} \cap \mathcal{M}^\perp = \{0\} \text{ and } \psi = \alpha\tilde{\psi}.$$

unique: $\frac{\langle \alpha\tilde{\psi}, \varphi\rangle}{\|\alpha\tilde{\psi}\|^2}\alpha\tilde{\psi} = \underbrace{\frac{\langle \tilde{\psi}, \varphi\rangle}{\|\tilde{\psi}\|^2}\tilde{\psi}}$ $\forall \alpha \in \mathbb{C}, \alpha \neq 0.$

Now we can uniquely decompose (with Theorem 3.15) any $\varphi = \varphi_{\mathcal{M}} + \varphi_{\mathcal{M}^\perp} = \varphi_{\mathcal{M}} + \frac{\langle \tilde{\psi}, \varphi\rangle}{\|\tilde{\psi}\|^2}\tilde{\psi}$ for any $\tilde{\psi} \in \mathcal{M}^\perp \setminus \{0\}$, and thus

$$T(\varphi) = T\left(\varphi_{\mathcal{M}} + \frac{\langle \tilde{\psi}, \varphi\rangle}{\|\tilde{\psi}\|^2}\tilde{\psi}\right) \overset{T(\varphi_{\mathcal{M}})=0}{=} \frac{\langle \tilde{\psi}, \varphi\rangle}{\|\tilde{\psi}\|^2} T(\tilde{\psi}) = \left\langle \frac{\overline{T(\tilde{\psi})}}{\|\tilde{\psi}\|^2}\tilde{\psi}, \varphi \right\rangle, \text{ i.e., } \psi_T = \frac{\overline{T(\tilde{\psi})}}{\|\tilde{\psi}\|^2}\tilde{\psi}. \quad \square$$

Riesz tells us that elements of $\mathcal{H}'$ can be be canonically identified with elements of $\mathcal{H}$:

**Corollary 3.40:**

$J: \mathcal{H} \to \mathcal{H}'$, $\psi \mapsto J\psi = \langle \psi, \cdot \rangle$ is a $\overbrace{\text{canonical}}^{\text{no arbitrary choices, e.g., of basis}}$ $\underbrace{\text{antilinear}}_{\text{due to antilinearity of the scalar product in the first variable}}$ $\overbrace{\text{bijection}}^{\text{by Riesz}}$ and a $\overbrace{\text{continuous}}^{\text{continuity of scalar product}}$ $\underbrace{\text{isometry}}_{\|J\psi\|_{\mathcal{L}(\mathcal{H},\mathbb{C})} = \|\psi\|_{\mathcal{H}}}$.

With that we can identify $A'$ canonically with an operator $A^*$ on $\mathcal{H}$:

**Definition 3.41:**

For $A \in \mathcal{L}(\mathcal{H})$, we define the $\boxed{\text{Hilbert space adjoint}}$ $A^*: \mathcal{H} \to \mathcal{H}$, $A^* = \overbrace{J^{-1}}^{\mathcal{H}' \to \mathcal{H}} \overbrace{A'}^{\mathcal{H}' \to \mathcal{H}'} \overbrace{J}^{\mathcal{H} \to \mathcal{H}'}$.

Sometimes $A^*$ is simply called "adjoint", or "Hermitian adjoint", and in the physics literature it is often denoted $A^\dagger$ ("A dagger").

Let us collect a few properties of $A^*$. First, with Riesz, we directly get

**Proposition 3.42:**

For $A \in \mathcal{L}(\mathcal{H})$ we have $\langle \psi, A\varphi \rangle = \langle A^*\psi, \varphi \rangle$ $\forall\, \psi, \varphi \in \mathcal{H}$ and this property uniquely determines $A^*$.

<u>Proof:</u> By the definitions we have

$$\langle \psi, A\varphi \rangle = (J\psi)(A\varphi) = A'(J\psi)(\varphi) = JJ^{-1}A'J\psi(\varphi) = JA^*\psi(\varphi) = \langle A^*\psi, \varphi \rangle.$$

Also, $\varphi \mapsto \langle \psi, A\varphi \rangle$ is continuous and linear, so due to Riesz there is a unique $\eta \in \mathcal{H}$ s.t. $\langle \psi, A\varphi \rangle = \langle \eta, \varphi \rangle$ $\forall \varphi \in \mathcal{H}$, so $\eta = A^*\psi$ is unique. □

Before we continue, a few more standard properties and an example

<u>Theorem 3.43:</u> For $A, B \in \mathcal{L}(\mathcal{H})$ and $\lambda \in \mathbb{C}$ we have

a) $(A+B)^* = A^* + B^*$ , $(\lambda A)^* = \bar{\lambda} A^*$

b) $(AB)^* = B^* A^*$

c) $\|A^*\| = \|A\|$

d) $A^{**} = A$

e) $\|AA^*\| = \|A^*A\| = \|A\|^2$

f) $\ker A = (\operatorname{im} A^*)^\perp$ and $\ker A^* = (\operatorname{im} A)^\perp$

<u>Proof:</u> HW (a), b), c) follow directly from definition, d), e), f) are short computations)

As an example, consider the left and right shifts on $\ell^2$:

The right shift is $T_r : \ell^2 \to \ell^2$, $(x_1, x_2, \dots) \mapsto (0, x_1, x_2, \dots)$. Then

$$\langle x, T_r y \rangle = \sum_{i=1}^{\infty} x_i (T_r y)_i = \sum_{i=2}^{\infty} x_i x_{i-1} = \sum_{i=1}^{\infty} x_{i+1} y_i =: \langle T_r^* x, y \rangle, \text{ so } T_r^* = T_\ell, \text{ where}$$

$T_\ell$ is the left shift $T_\ell : \ell^2 \to \ell^2$, $(x_1, x_2, \dots) \mapsto (x_2, x_3, \dots)$.

Note that $T_r$ is isometric ($\|Tx\| = \|x\|$), but not surjective, so it is not unitary.

We have $T_r^* T_r = \mathrm{id}$, but $T_r T_r^* \neq \mathrm{id}$, so $T_r^*$ is not the inverse of $T_r$ (which isn't even invertible).

Based on this example, let us make the following nice connection to unitary operators:

**Proposition 3.45:** $U \in \mathcal{L}(\mathcal{H})$ is $\underbrace{\text{unitary}}_{\text{surjective + isometric}}$ if and only if $U^* = U^{-1}$.

**Proof:** Next time.