Last time: • $(T, \mathcal{D}(T))$ a densly def. lin. op. on $\mathcal{H}$.

• $\mathcal{D}(T^*) := \{ \psi \in \mathcal{H} : \exists \eta \in \mathcal{H} \text{ s.t. } \langle \psi, T\varphi \rangle = \langle \eta, \varphi \rangle \ \forall \varphi \in \mathcal{D}(T) \}$

• η is uniquely determined and $T^* : \mathcal{D}(T^*) \to \mathcal{H}, \psi \mapsto \eta$ is the adjoint op.

• $(T, \mathcal{D}(T))$ is self-adjoint if $\mathcal{D}(T) = \mathcal{D}(T^*)$ and $T = T^*$.

Let us exemplify the definitions with $-i\frac{d}{dx}$ on $[0,1]$ again.

↳ First, consider $(D_{min}, \mathcal{D}(D_{min}))$, $D_{min} = -i\frac{d}{dx}$, with $\mathcal{D}(D_{min}) = \{ \varphi \in H^1[0,1] : \varphi(0) = 0 = \varphi(1) \}$

Then $\forall \varphi \in \mathcal{D}(D_{min})$:

integration by parts, boundary terms vanish on $\mathcal{D}(D_{min})$

$$\langle \psi, D_{min} \varphi \rangle = \int_0^1 \overline{\psi(x)} \left(-i\frac{d}{dx}\varphi(x)\right) dx = \int_0^1 \overline{\left(-i\frac{d}{dx}\psi(x)\right)} \varphi(x) dx = \langle \underbrace{-i\frac{d}{dx}\psi}_{=\eta}, \varphi \rangle,$$

which works for all $\frac{d\psi(x)}{dx} \in L^2$, i.e., $\psi \in H^1([0,1])$.

So $\mathcal{D}(D^*_{min}) = H^1([0,1]) \neq \mathcal{D}(D_{min})$, and $(D_{min}, \mathcal{D}(D_{min}))$ is not self-adjoint.

↳ For $D_\theta = -i\frac{d}{dx}$ with $\mathcal{D}(D_\theta) = \{ \varphi \in H^1([0,1]) : e^{i\theta}\varphi(1) = \varphi(0) \}$ we find $\forall \varphi \in \mathcal{D}(D_\theta)$:

$$\langle \psi, D_\theta \varphi \rangle = i\left(\overline{\psi(0)}\varphi(0) - \overline{\psi(1)}\varphi(1)\right) + \langle \underbrace{-i\frac{d}{dx}\psi}_{\eta}, \varphi \rangle,$$ so in order to get

$\langle \psi, D_\theta \varphi \rangle = \langle \eta, \varphi \rangle$ we need $\psi \in H^1([0,1])$ <u>and</u> $\overline{\psi(0)}\varphi(0) = \overline{\psi(1)}\varphi(1) \iff$

$$\frac{\overline{\psi(0)}}{\overline{\psi(1)}} = \frac{\varphi(1)}{\varphi(0)} = e^{-i\theta} \text{ (by def.)}, \text{ i.e., } \psi(0) = e^{i\theta}\psi(1).$$

So $\mathcal{D}(D^*_\theta) = \mathcal{D}(D_\theta)$ and $D^*_\theta = D_\theta$, so D_θ is self-adjoint.

A big result is that indeed the following holds:

Theorem 3.58: A densly defined operator $(H, D(H))$ is generator of a strongly continuous unitary group $U(t)$ if and only if it is self-adjoint.

"$\Leftarrow$" follows from the spectral theorem; "$\Rightarrow$" is Stone's theorem

We skip the proofs here.

A few examples:

- $H_0 = -\frac{\Delta}{2}$ with domain $H^2(\mathbb{R}^d)$ is self-adjoint

- $H_0 = -\frac{\Delta}{2}$ with domain $C_0^\infty(\mathbb{R}^3 \setminus \{0\})$ is not self-adjoint, but has several self-adjoint extensions: H_0 with domain $H^2(\mathbb{R}^3)$ is one of them, but there are others corresponding to δ-interaction at the origin, i.e., formally $H_0 = -\frac{\Delta}{2} + \delta(x)$

- $H_{Dirac} = -i\hbar c \vec{\alpha} \cdot \vec{\nabla} + mc^2 \beta$ with domain $C_0^\infty(\mathbb{R}^3 \setminus \{0\} \to \mathbb{C}^4)$ is not self-adjoint, but

  $\hookrightarrow \beta \in M(\mathbb{C}^4 \times \mathbb{C}^4)$

  $\vec{\alpha} = (\alpha_1, \alpha_2, \alpha_3)$ with $\alpha \in M(\mathbb{C}^4 \times \mathbb{C}^4)$

  has only one self-adjoint extension: H_{Dirac} with domain $H^1(\mathbb{R}^3 \to \mathbb{C}^4)$

  $\Rightarrow$ In 3-dim., point-interaction exists in non-rel. QM, but not in rel. QM

- $H_{Dirac} - \frac{\lambda}{|x|}$ is self-adjoint on $H^1(\mathbb{R}^3 \to \mathbb{C}^4)$ only for λ small enough

Finally, let us state Kato-Rellich. The idea is that we often consider operators $H = H_0 + V$, where we know that H_0 is self-adjoint with domain $\mathcal{D}(H_0)$ and we want to know whether also H is self-adjoint. Often the V's are such that they do not disturb $\mathcal{D}(H_0)$ too much, i.e., they do not introduce new boundary points.

<u>Definition:</u> Let A, B be densly defined linear operators with $\mathcal{D}(A) \subset \mathcal{D}(B)$ and such that there are $a, b \geq 0$ s.t.

$\|B\varphi\| \leq a\|A\varphi\| + b\|\varphi\| \;\; \forall \varphi \in \mathcal{D}(A)$. Then B is called relatively bounded by A (or A-bounded), and the infimum over all permissible a is called the relative bound.

If the relative bound $= 0$, B is called infinitesimally A-bounded.

<u>Theorem (Kato-Rellich):</u> Let A be self-adjoint, B symmetric and A-bounded with relative bound $a < 1$. Then $A + B$ is self-adjoint on $\mathcal{D}(A+B) = \mathcal{D}(A)$.

Ex.: • $A = -\Delta$, $B = M_V$ with $V = V_1 + V_2$ with $V_1 \in L^2(\mathbb{R}^3)$, $V_2 \in L^\infty(\mathbb{R}^3)$

Then B is infinitesimally $(-\Delta)$-bounded, and $H = -\Delta + V$ is self-adjoint, see HW 11.

Note: This implies $H = -\Delta + \frac{1}{|x|}$ is self-adjoint on $\mathcal{D}(H) = H^2(\mathbb{R}^3)$, see HW 11.

Another approach to proving existence of self-adjoint extensions is via the Friedrichs extension. The advantage of this approach is that it is very easy to apply in practice. The disadvantage is that it only gives existence of a self-adjoint extension, and it does not provide information on its domain, nor uniqueness.

It uses the following definition:

Definition 3.114: An operator H is called semibounded if there is a $c \in \mathbb{R}$ s.t. for all $\psi \in \mathcal{D}(H)$, $\langle \psi, H\psi \rangle \geq c\|\psi\|^2$ (from below) or $\langle \psi, H\psi \rangle \leq c\|\psi\|^2$ (from above).

Then we have:

Theorem 3.115 (Friedrichs extension):

Any densly defined semibounded operator H has a self-adjoint extension, which satisfies the same upper/lower bound.

We skip the proof.

E.g., for $-\Delta$ on $C_0^\infty(\Omega)$, $\Omega \subset \mathbb{R}^d$ open, we find $\langle \varphi, (-\Delta)\varphi \rangle = \|\nabla\varphi\|^2 \geq 0 \cdot \|\varphi\|^2$,

so $(-\Delta, C_0^\infty(\Omega))$ has a self-adjoint extension. Same for $-\Delta + V$ with $V \geq 0$.

One can actually define one particular self-adjoint extension uniquely via quadratic forms. This is then called <u>the</u> Friedrichs extension.

(E.g., for $H = -\frac{d^2}{dx^2}$ on $C_0^\infty((0,1))$, the Friedrichs extension is the Dirichlet Laplacian. But there are other extensions with other boundary conditions, e.g., Neumann.)