Last time we defined the orthogonal projectors $p^\varphi: L^2 \to L^2$, $\chi \mapsto \langle \varphi, \chi \rangle \varphi$ and $q^\varphi = \mathbb{1} - p^\varphi$. p_j^φ and q_j^φ denote the projections in the j-th coordinate only.

Note that in "bra-ket" notation, we write $p^\varphi |\chi\rangle = |\varphi\rangle\langle\varphi|\chi\rangle$, i.e., $p^\varphi = |\varphi\rangle\langle\varphi|$.

p^φ tells us "how much" of the wave function is in the state φ. E.g., if $\varphi^\perp$ is orthogonal to φ, i.e., $\langle \varphi^\perp, \varphi \rangle = 0$, and $\chi := \frac{a}{\sqrt{a^2+b^2}} \varphi + \frac{b}{\sqrt{a^2+b^2}} \varphi^\perp$ s.t. $\|\chi\|^2 = \frac{a^2+b^2}{a^2+b^2} = 1$, then $\|p^\varphi \chi\| = \frac{a}{\sqrt{a^2+b^2}}$.

Next, we define a projection that measures "how much" of the N-body wave function is not in the state φ.

Definition 4.3: For $\varphi \in L^2$, $\|\varphi\| = 1$, we def. for any $0 \le k \le N$ the bounded operator

$$P_{N,k}^\varphi : L^2(\mathbb{R}^{3N}) \to L^2(\mathbb{R}^{3N}),\quad P_{N,k} := \sum_{\vec{a} \in A_k} \prod_{j=1}^N (p_j^\varphi)^{1-a_j} (q_j^\varphi)^{a_j}, \text{ where}$$

$$A_k := \Big\{ \vec{a} \in \{0,1\}^N : \sum_{j=1}^N a_j = k \Big\}.$$

Example:

- $P_{N,0}^\varphi = p_1^\varphi \cdots p_N^\varphi$
- $P_{N,1}^\varphi = q_1^\varphi p_2^\varphi \cdots p_N^\varphi + p_1^\varphi q_2^\varphi p_3^\varphi \cdots p_N^\varphi + \ldots + p_1^\varphi \cdots p_{N-1}^\varphi q_N^\varphi = \sum_{m=1}^N q_m^\varphi \prod_{\substack{j=1 \\ j \ne m}}^N p_j^\varphi$
- $P_{N,N}^\varphi = q_1^\varphi \cdots q_N^\varphi$

Note: $P_{N,k}$ contains in each summand k q's and $(N-k)$ p's

Lemma 4.4: We have (i) $P^{\varphi}_{N,k}$ is an orthogonal projector for all $0 \le k \le N$,

(ii) $P^{\varphi}_{N,k} P^{\varphi}_{N,j} = 0$ for all $j \neq k$,

(iii) $\sum_{k=0}^{N} P^{\varphi}_{N,k} = \mathbb{1}$

(iv) $\sum_{k=0}^{N} \frac{k}{N} P^{\varphi}_{N,k} = \frac{1}{N} \sum_{j=1}^{N} q^{\varphi}_j$

Proof: HW 10.

We can now decompose the wave function as $\Psi_N = \sum_{k=0}^{N} P^{\varphi}_{N,k} \Psi_N$.

Then $\| P^{\varphi}_{N,k} \Psi_N \|^2$ is the probability for k particles not being in the state φ.

Thus, we define:

Definition 4.5: The expected relative number of particles not in the state φ

is given by $\alpha(\Psi_N, \varphi) := \sum_{k=0}^{N} \frac{k}{N} \| P^{\varphi}_{N,k} \Psi_N \|^2$.

Corollary 4.6: For all symmetric $\Psi_N \in L^2(\mathbb{R}^{3N})$ and for all $\varphi \in L^2(\mathbb{R}^3)$,

$$\alpha(\Psi_N, \varphi) = \sum_{k=0}^{N} \frac{k}{N} \langle \Psi_N, P^{\varphi}_{N,k} \Psi_N \rangle = \frac{1}{N} \sum_{j=1}^{N} \langle \Psi_N, q^{\varphi}_j \Psi_N \rangle = \langle \Psi_N, q^{\varphi}_1 \Psi_N \rangle = \| q^{\varphi}_1 \Psi_N \|^2.$$

Proof: First equality follows directly from Lemma 4.4 (i), second from (iv) and third from Ψ_N symmetric.

Note:
- For $\Psi_N = \prod_{i=1}^{N} \varphi$, we have $\alpha(\Psi_N, \varphi) = \| q^{\varphi}_1 \prod_{i=1}^{N} \varphi \|^2 = 0$.
- For $\varphi^{\perp}$ with $\langle \varphi^{\perp}, \varphi \rangle = 0$, $\Psi_N = \prod_{i=1}^{N} \varphi^{\perp}$, we have $\alpha(\Psi_N, \varphi) = 1$.
- $0 \le \alpha(\Psi_N, \varphi) \le 1$ for $\|\Psi_N\| = 1 = \|\varphi\|$
- For $\Psi_N = \frac{1}{\sqrt{N}} \sum_{j=1}^{N} \prod_{\substack{i=1 \\ i \neq j}}^{N} \varphi(x_i) \varphi_{\perp}(x_j)$, we have $\alpha(\Psi_N, \varphi) = \frac{1}{N}$ (but $\| \Psi_N - \prod_{i=1}^{N} \varphi(x_i) \|_{L^2(\mathbb{R}^{3N})} = 2$).