Last time we introduced the reduced one-particle density matrix $\gamma_{\Psi_N}: L^2 \to L^2$ via its kernel

$$\gamma_{\Psi_N}(x,y) := \int dx_2 \dots dx_N \, \overline{\Psi_N(y, x_2, \dots, x_N)} \, \Psi_N(x, x_2, \dots, x_N).$$

We had also defined the expected number of particles not in the product state $\Pi\varphi$ as

$$\alpha(\Psi_N, \varphi) = \sum_{k=0}^{N} \frac{k}{N} \langle \Psi_N, P^{\varphi}_{N,k} \Psi_N \rangle = \langle \Psi_N, q^{\varphi}_1 \Psi_N \rangle.$$

Next, we establish a relation between $\gamma_{\Psi_N} - \overbrace{\gamma_{\prod_{i=1}^{N} \varphi}}^{= p^{\varphi}}$ and $\alpha(\Psi_N, \varphi)$:

Lemma 4.10: For any symmetric $\Psi_N \in L^2(\mathbb{R}^{3N})$, $\varphi \in L^2(\mathbb{R}^3)$ with $\|\Psi_N\| = 1 = \|\varphi\|$, we have

(a) $\alpha(\Psi_N, \varphi) \leq \|\gamma_{\Psi_N} - p^{\varphi}\|_{S_1} \leq 4\sqrt{\alpha(\Psi_N, \varphi)}$,

(b) $|\langle \Psi_N, A_1 \Psi_N \rangle - \langle \varphi, A\varphi \rangle| \leq \|A\| \, 4\sqrt{\alpha(\Psi_N, \varphi)}$ for any $A \in S_\infty(L^2(\mathbb{R}^3))$.

Proof: (a) Note that $p^{\varphi} + q^{\varphi} = \mathbb{1}$ by def., so we can decompose

$$\gamma_{\Psi_N} = (p^{\varphi} + q^{\varphi}) \gamma_{\Psi_N} (p^{\varphi} + q^{\varphi})$$

$$= \underbrace{p^{\varphi} \gamma_{\Psi_N} p^{\varphi}}_{\substack{= p^{\varphi} \langle \varphi, \gamma_{\Psi_N} \varphi \rangle \\ = p^{\varphi} \langle \Psi_N, p^{\varphi}_1 \Psi_N \rangle = p^{\varphi} - p^{\varphi} \langle \Psi_N, q^{\varphi}_1 \Psi_N \rangle}} + p^{\varphi} \gamma_{\Psi_N} q^{\varphi} + q^{\varphi} \gamma_{\Psi_N} p^{\varphi} + q^{\varphi} \gamma_{\Psi_N} q^{\varphi}$$

$$\Rightarrow \|\gamma_{\Psi_N} - p^\varphi\|_{\mathfrak{S}} = \| - p^\varphi \langle \Psi_N, q_1^\varphi \Psi_N\rangle + p^\varphi \gamma_{\Psi_N} q^\varphi + q^\varphi \gamma_{\Psi_N} p^\varphi + q^\varphi \gamma_{\Psi_N} q^\varphi \|_{\mathfrak{S}}$$

$$\leq \underbrace{\|p^\varphi\|_{\mathfrak{S}}}_{\leq 1} \underbrace{\langle \Psi_N, q_1^\varphi \Psi_N\rangle}_{\substack{= \|q_1^\varphi \Psi_N\|^2 \\ = \alpha(\Psi_N, \varphi)}} + \underbrace{\|p^\varphi \gamma_{\Psi_N} q^\varphi\|_{\mathfrak{S}}}_{\leq \underbrace{\|p^\varphi \Psi_N\|}_{\leq 1} \underbrace{\|q^\varphi \Psi_N\|}_{= \sqrt{\alpha(\Psi_N, \varphi)}} \text{ as in proof of Lemma 4.9}} + \|q^\varphi \gamma_{\Psi_N} p^\varphi\|_{\mathfrak{S}} + \|q^\varphi \gamma_{\Psi_N} q^\varphi\|_{\mathfrak{S}}$$

$$\leq 2 \underbrace{\alpha(\Psi_N, \varphi)}_{\leq \sqrt{\alpha(\Psi_N,\varphi)} \text{ since } 0 \leq \alpha(\Psi_N,\varphi) \leq 1} + 2\sqrt{\alpha(\Psi_N, \varphi)}$$

$$\leq 4\sqrt{\alpha(\Psi_N, \varphi)}$$

Also:

$$\alpha(\Psi_N, \varphi) = \|q_1^\varphi \Psi_N\|^2 = 1 - \|p_1^\varphi \Psi_N\|^2 = 1 - \langle \varphi, \gamma_{\Psi_N} \varphi\rangle = \langle \varphi, (p^\varphi - \gamma_{\Psi_N})\varphi\rangle \leq \|p^\varphi - \gamma_{\Psi_N}\|_{\mathfrak{S}}.$$

(b) Similarly, we find for any $A \in \mathfrak{S}(L^2(\mathbb{R}^3))$:

$$\langle \Psi_N, A_1 \Psi_N\rangle - \langle \varphi, A\varphi\rangle$$

$$= \langle \Psi_N, \overbrace{(p_1^\varphi + q_1^\varphi)}^{=\mathbb{1}} A_1 (p_1^\varphi + q_1^\varphi)\Psi_N\rangle - \langle \varphi, A\varphi\rangle$$

$$= \underbrace{\langle \Psi_N, \underbrace{p_1^\varphi A_1 p_1^\varphi}_{\substack{= |\varphi\rangle\langle\varphi|A|\varphi\rangle\langle\varphi| \\ = \langle\varphi, A\varphi\rangle p_1^\varphi}} \Psi_N\rangle - \langle \varphi, A\varphi\rangle}_{\substack{= \langle\varphi, A\varphi\rangle(\langle\Psi_N, p_1^\varphi \Psi_N\rangle - 1) \\ = -\langle\varphi, A\varphi\rangle\, \alpha_N(\Psi_N, \varphi)}} + \underbrace{\langle \Psi_N, p_1^\varphi A_1 q_1^\varphi \Psi_N\rangle}_{\substack{\leq \|A\| \, \|p_1^\varphi \Psi_N\| \, \|q_1^\varphi \Psi_N\| \\ \leq \|A\| \sqrt{\alpha(\Psi_N,\varphi)}}} + \langle \Psi_N, q_1^\varphi A_1 p_1^\varphi \Psi_N\rangle + \underbrace{\langle \Psi_N, q_1^\varphi A_1 q_1^\varphi \Psi_N\rangle}_{\leq \|A\| \, \alpha(\Psi_N,\varphi)}$$

Cauchy-Schwarz

$$\Rightarrow |\langle \Psi_N, A_1 \Psi_N\rangle - \langle \varphi, A\varphi\rangle| \leq \|A\| \, 4\sqrt{\alpha(\Psi_N, \varphi)} \qquad \square$$

Conclusion: If we can show that $\alpha(\Psi_N(t), \varphi(t)) \xrightarrow{N\to\infty} 0$, then also $\gamma_{\Psi_N} \xrightarrow{N\to\infty} p^\varphi$ and $\langle \Psi_N, A_1 \Psi_N\rangle \xrightarrow{N\to\infty} \langle \varphi, A\varphi\rangle$ $\forall A \in \mathfrak{S}$.