Constructor University
Fall 2023

November 9, 2023

Operations Research

Midterm Exam

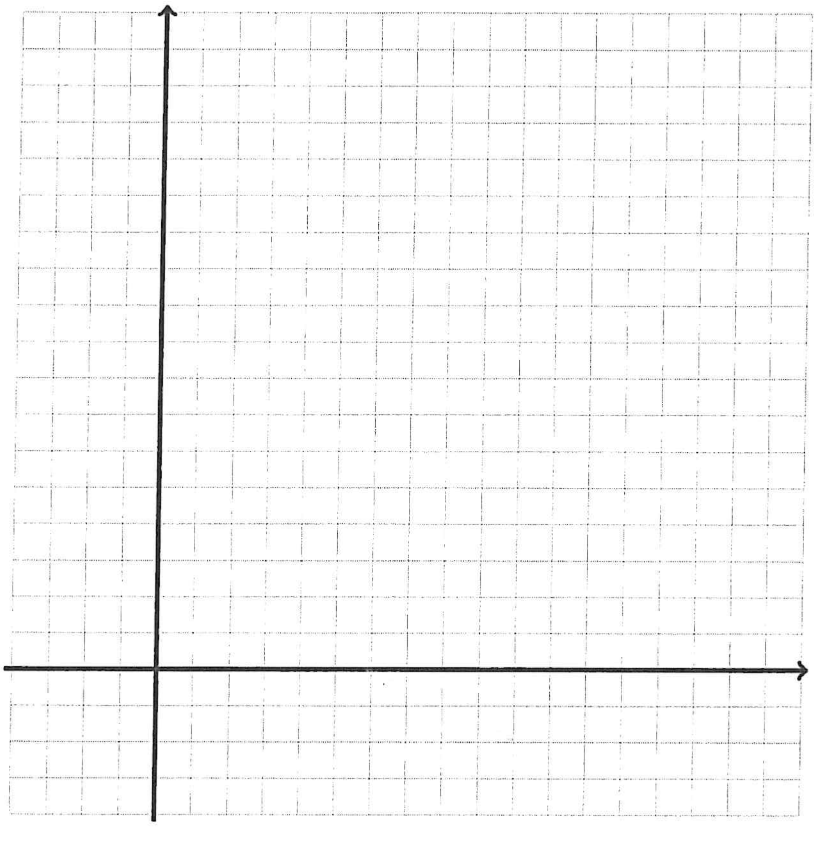
(For bonus points only.)

Instructions:

- Do all the work on this exam paper.
- Show your work, i.e., carefully write down the steps of your solution. You will receive points not just based on your final answer, but on the correct steps in your solution.
- No tools or other resources are allowed for this exam. In particular, no notes and no calculators.

Name: Solutions

Matric. No.: ______________________



Problem 1: Graphical Method [25 points]
Consider the following Linear Programming (LP) problem LP1: Maximize

$$Z = 2x_1 + x_2$$

subject to

$$\begin{aligned} 3x_1 - x_2 &\leq 6, \\ \frac{1}{2}x_1 + x_2 &\leq 4, \\ x_1, x_2 &\geq 0. \end{aligned}$$

(a) Solve the problem with the graphical method only, i.e., draw the feasible region, and read off the optimal x_1, x_2 (as far as the accuracy of the picture permits), and the optimal Z.

(b) From part (a), identify the two binding constraints. Using this information, compute the optimal solution and the corresponding value of Z exactly.

(c) Now consider another Linear Programming problem, LP2: Minimize

$$\widetilde{Z} = 6y_1 + 4y_2$$

subject to

$$\begin{aligned} 3y_1 + \frac{1}{2}y_2 &\geq 2, \\ -y_1 + y_2 &\geq 1, \\ y_1, y_2 &\geq 0. \end{aligned}$$

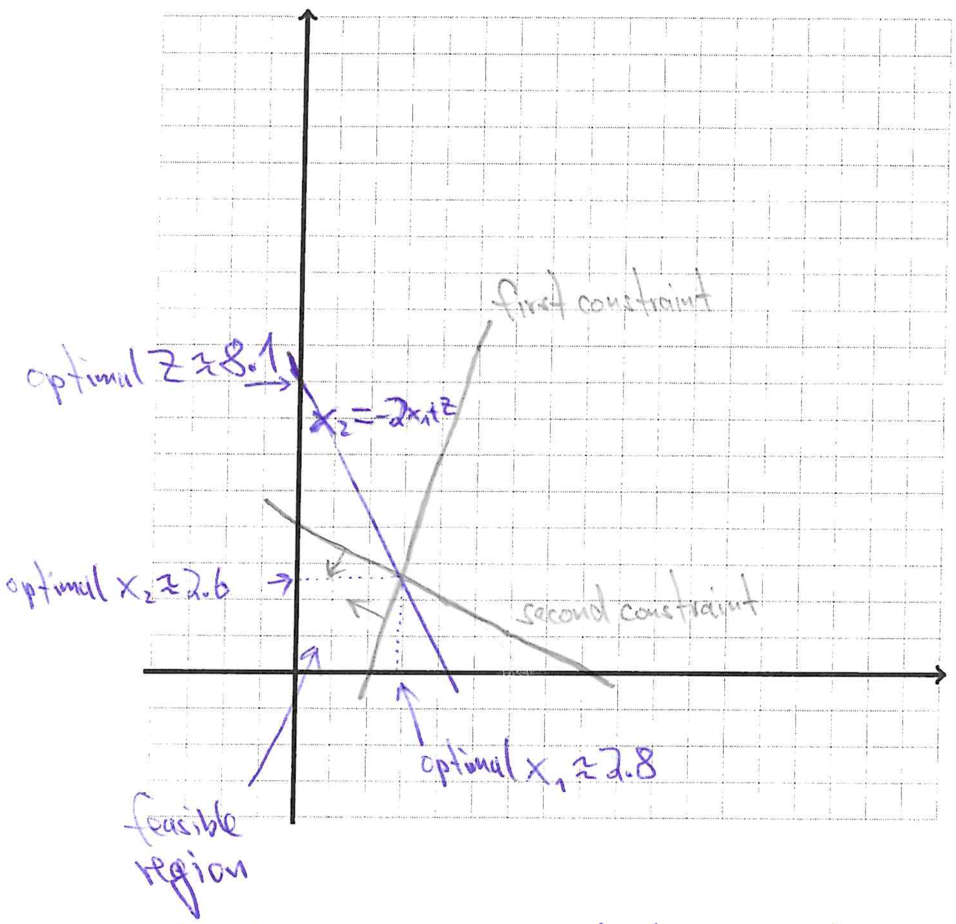
Solve this LP problem as well with the graphical method, and compute the optimal y_1, y_2 and $\widetilde{Z}$.

(d) What is the meaning and interpretation of problem LP2 from (c) compared to the original LP problem LP1 above?

(e) Consider again LP1 above. Suppose we increase the right-hand side of the first constraint from 6 to 7. *Using only parts (c) and (d)*, determine the new optimal Z.

Problem 1: Extra Space

(6) a)



(5) b) The two constraints $3x_1 - x_2 \leq 6$ and $\frac{1}{2}x_1 + x_2 \leq 4$ are binding, thus the optimal solution is at the intersection of

$3x_1 - x_2 = 6$ and $\frac{1}{2}x_1 + x_2 = 4$

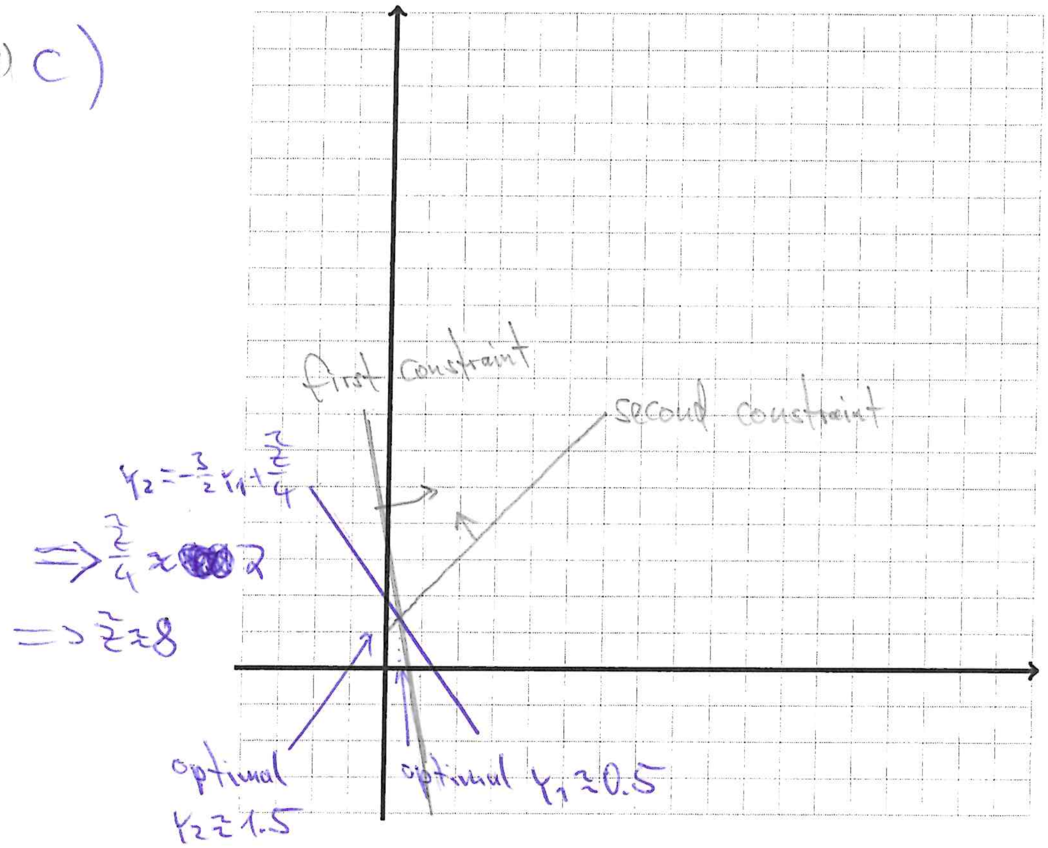
$\Rightarrow \frac{7}{2}x_1 = 10 \Rightarrow x_1 = \frac{20}{7} \Rightarrow x_2 = 4 - \frac{10}{7} = \frac{18}{7}$

$\Rightarrow Z = 2 \cdot \frac{20}{7} + \frac{18}{7} = \frac{58}{7} = 8 + \frac{2}{7}$

Those values agree (within reasonable accuracy) with the values from a)

Problem 1: Extra Space

(8) c)



Optimal solution is where $3y_1 + \frac{1}{2}y_2 = 2$ and $-y_1 + y_2 = 1$ intersect.

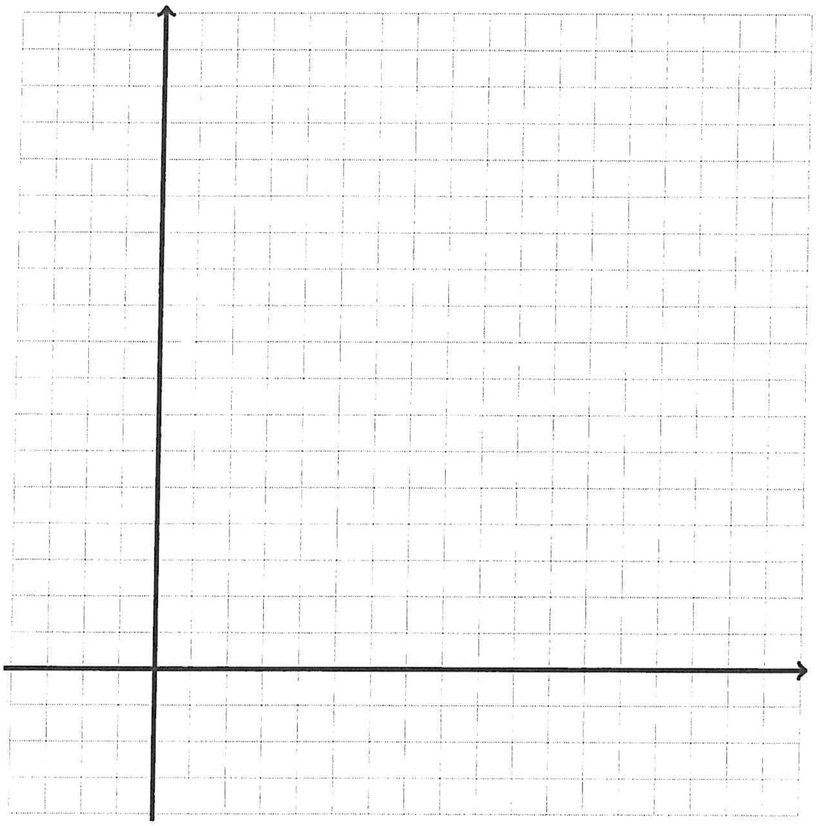
$\Rightarrow \frac{7}{2}y_1 = \frac{3}{2} \Rightarrow y_1 = \frac{3}{7} \Rightarrow y_2 = 1 + \frac{3}{7} = \frac{10}{7}$

$\Rightarrow z = 6 \cdot \frac{3}{7} + 4 \cdot \frac{10}{7} = \frac{58}{7}$

(3) d) LP2 is the dual problem to LP1.
Solving the dual gives us the shadow prices y_1, y_2, which tell us how z changes if we change the capacities in the constraints (by a small amount).

e) Increasing from 6 to 7 means the objective fct. will increase by $1 \cdot y_1^{opt} = \frac{3}{7}$. Thus, the new optimal z is $\frac{58}{7} + \frac{3}{7} = \frac{61}{7}$.

Problem 1: Extra Space



Problem 2: Standard Form and Simplex Method [25 points]

(a) Consider the following Linear Programming problem: Maximize

$$Z = 2x_1 + x_2$$

subject to

$$4x_1 + x_2 \leq 4,$$
$$2x_1 + 3x_2 \leq 6,$$
$$x_1, x_2 \geq 0,$$

Write this problem in standard form, i.e., as the problem to minimize

$$\tilde{Z} = c^{\mathrm{T}} \tilde{x}$$

subject to

$$A\tilde{x} = b, \quad \tilde{x} \geq 0.$$

Specify exactly the vectors b, c and the matrix A.

(b) Solve the problem with the simplex method as shown in class. *(Note: You will not receive points for simply stating the solution, but only for using the simplex method step by step.)*

5) a) Standard form: Minimize $\tilde{Z} = -2x_1 - x_2$

subject to $4x_1 + x_2 + s_1 = 4$

$2x_1 + 3x_2 + s_2 = 6$

with $x_1, x_2, s_1, s_2 \geq 0$.

$\Rightarrow b = \begin{pmatrix} 4 \\ 6 \end{pmatrix}, c = \begin{pmatrix} -2 \\ -1 \\ 0 \\ 0 \end{pmatrix}, A = \begin{pmatrix} 4 & 1 & 1 & 0 \\ 2 & 3 & 0 & 1 \end{pmatrix}$

20)

b) Simplex tableau:

4	1	1	0	4
2	3	0	1	6
-2	-1	0	0	0

$R_1/4$

$\Rightarrow -\frac{R_1}{2} + R_2$

1	$\frac{1}{4}$	$\frac{1}{4}$	0	1
0	$\frac{5}{2}$	$-\frac{1}{2}$	1	4
0	$-\frac{1}{2}$	$\frac{1}{2}$	0	2

$\Rightarrow$

$-\frac{1}{4}\cdot\frac{2}{5}R_2 + R_1$

$\frac{2}{5}\cdot R_2$

$\frac{1}{5}R_2 + R_3$

1	0	$\frac{4}{10}$	$-\frac{1}{10}$	$\frac{3}{5}$
0	1	$-\frac{1}{5}$	$\frac{2}{5}$	$\frac{8}{5}$
0	0	$\frac{2}{5}$	$\frac{1}{5}$	$\frac{14}{5}$

⇑

The optimal solution is $x_1 = \frac{3}{5}, x_2 = \frac{8}{5}$, with $Z = \frac{14}{5}$

$(\tilde{Z} = -\frac{14}{5})$

Problem 2: Extra Space

Problem 2: Extra Space

Problem 2: Extra Space

Problem 3: Network optimization [25 points]

A company has 3 factories that produce goods that need to be delivered to four warehouses. The available supply s_i at factory i, the demand d_j at warehouse j, and the shipping costs c_{ij} to ship from factory i to warehouse j are as in the following table:

	Warehouse 1	2	3	4	Supply
Factory 1	$500	$400	$600	$700	10
Factory 2	$600	$600	$400	$500	15
Factory 3	$300	$500	$700	$600	15
Demand	5	10	10	15	

We are looking for the most cost effective way to ship all the supply from the factories to the warehouses.

(5) (a) Draw a network representation of the problem.

(10) (b) Formulate this transportation problem as a Linear Programming problem, i.e., introduce appropriate decision variables, write down the objective function (and in which sense it is to be optimized) and the constraints.

(5) (c) Are there feasible solutions to this LP problem for the data given in the table? Briefly explain your answer.

(5) (d) Determine which of the following is the optimal solution to this LP problem. (Note that (i, j) means the amount shipped from factory i to warehouse j.) Clearly explain why you exclude certain possibilities!

```
Solution 1:
{(1, 1): 4.5,
 (1, 2): 5.5,
 (1, 3): 0.0,
 (1, 4): 0.0,
 (2, 1): 0.0,
 (2, 2): 0.0,
 (2, 3): 0.0,
 (2, 4): 15.0,
 (3, 1): 0.5,
 (3, 2): 4.5,
 (3, 3): 10.0,
 (3, 4): 0.0}
```

Cannot be optimal, because all x_{ij} in the optimal sol. must be integer!

```
Solution 2:
{(1, 1): 5.0,
 (1, 2): 5.0,
 (1, 3): 0.0,
 (1, 4): 0.0,
 (2, 1): 0.0,
 (2, 2): 0.0,
 (2, 3): 0.0,
 (2, 4): 15.0,
 (3, 1): 0.0,
 (3, 2): 5.0,
 (3, 3): 10.0,
 (3, 4): 0.0}
```

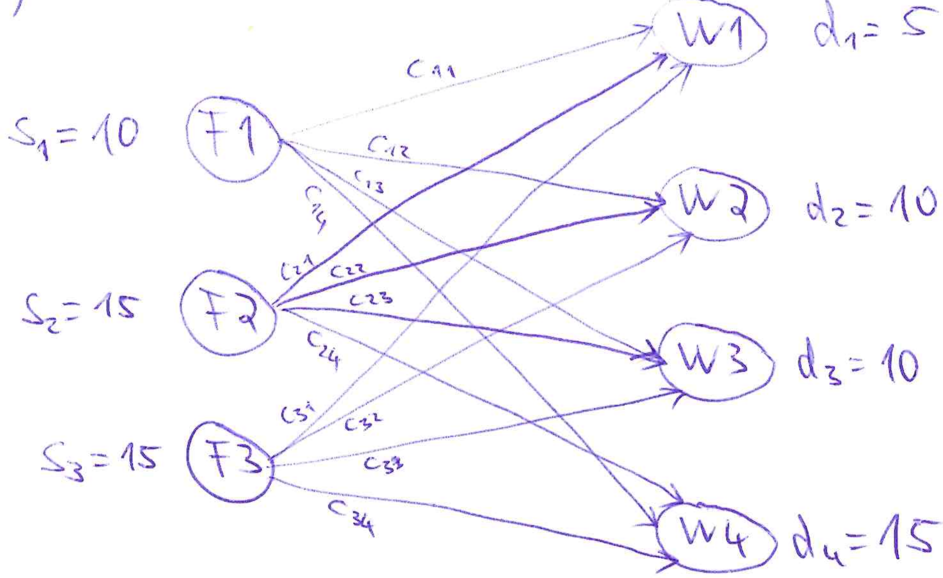
This is the only possibility.

```
Solution 3:
{(1, 1): 5.0,
 (1, 2): 10.0,
 (1, 3): 0.0,
 (1, 4): 0.0,
 (2, 1): 5.0,
 (2, 2): 0.0,
 (2, 3): 0.0,
 (2, 4): 10.0,
 (3, 1): 0.0,
 (3, 2): 5.0,
 (3, 3): 10.0,
 (3, 4): 5.0}
```

Cannot be optimal, since, e.g. 5+10 units are shipped from factory 1, although only 10 units of supply are available

Problem 3: Extra Space

a) c_{ij} as in table



b) Decision variables: x_{ij} = # of units shipped from factory i to warehouse j.

$i = 1,2,3$, $j = 1,2,3,4$

Objective: Minimize cost $z = \sum_{i=1}^{3} \sum_{j=1}^{4} c_{ij} x_{ij}$

Constraints: $\sum_{j=1}^{4} x_{ij} = S_i \quad \forall i = 1,2,3$

$\sum_{i=1}^{3} x_{ij} = d_j \quad \forall j = 1,2,3,4$

and $x_{ij} \geq 0$

c) Yes, there is even always an optimal solution if $\sum_{i=1}^{3} S_i = \sum_{j=1}^{4} d_j$ (supply = demand). Here, supply $= 10+15+15 = 40$,

demand $= 5+10+10+15 = 40$.

Problem 3: Extra Space

Problem 3: Extra Space

Problem 4: Pyomo [25 points]

The pyomo program on the last exam page shows an implementation of a minimum cost flow problem (which we discussed in class).

(6) (a) Draw the corresponding network representation of the problem. (Make sure to include the given values of the parameters from the pyomo program.)

(3) (b) Write down the Linear Programming problem that is solved in mathematical notation (i.e., write down the objective function and the constraints).

(5) (c) Briefly explain the meaning of the parameters b, C, U in the context of minimum cost flow problems. In particular, explain what positive, negative, or zero values for entries in b mean.

(3) (d) Suppose one of the entries in U can be slightly increased. Which one should we choose in order to lower the costs? (You will receive points only if you justify your answer correctly, and not for guessing.)

(3) (e) Suppose one of the entries F1 or F2 in b is increased by two units, and simultaneously one of the entries W1 or W2 in b is decreased by two units. Which entries should be increased/decreased in order to be most cost effective? What will be the extra cost for the increase/decrease?

(3) (f) Management notices that there is some flexibility between F1 and F2 concerning the units of b, i.e., F1 could be increased by a few units while simultaneously decreasing F2 by the same number of units, or the other way around. Which of the following three solutions should we recommend in order to be most cost effective:

1. Increase F1 by a few units and decrease F2.
2. Decrease F1 by a few units and increase F2.
3. Do not change the values of F1 and F2.

Explain your answer. (You will not receive points for a random choice.)

See Fall 2022 [illegible] exam solution.

midterm make-up

Problem 4: Extra Space

Problem 4: Extra Space

Problem 4: Extra Space

```
from pyomo.environ import *
from pyomo.opt import *
opt = solvers.SolverFactory("glpk")
```

```
b = {'F1':50,
     'F2':40,
     'DC':0,
     'W1':-30,
     'W2':-60}

C = {('F1','F2'):200,
     ('F1','DC'):400,
     ('F1','W1'):900,
     ('F2','DC'):300,
     ('DC','W2'):100,
     ('W1','W2'):300,
     ('W2','W1'):200}

U = {('F1','F2'):10,
     ('DC','W2'):80}

N = list(b.keys())
A = list(C.keys())
V = list(U.keys())
```

```
model = ConcreteModel()
model.x = Var(A, within=NonNegativeReals)
```

```
def flow_rule(model, n):
    InFlow  = sum(model.x[i,j] for (i,j) in A if j==n)
    OutFlow = sum(model.x[i,j] for (i,j) in A if i==n)
    return OutFlow - InFlow == b[n]

model.flow = Constraint(N, rule=flow_rule)
```

```
def capacity_rule(model, i, j):
    return model.x[i,j] <= U[i,j]

model.capacity = Constraint(V, rule=capacity_rule)
```

```
model.cost = Objective(expr = sum(model.x[a]*C[a] for a in A), sense=minimize)
```

```
model.dual = Suffix(direction=Suffix.IMPORT)
results = opt.solve(model)
model.x.get_values()
```

```
{('F1', 'F2'): 0.0,
 ('F1', 'DC'): 40.0,
 ('F1', 'W1'): 10.0,
 ('F2', 'DC'): 40.0,
 ('DC', 'W2'): 80.0,
 ('W1', 'W2'): 0.0,
 ('W2', 'W1'): 20.0}
```

```
model.cost.expr()
```

```
49000.0
```

```
for n in N:
    print(n, model.dual[model.flow[n]])
```

```
F1 700.0
F2 600.0
DC 300.0
W1 -200.0
W2 0.0
```

```
for j in V:
    print(j, model.dual[model.capacity[j]])
```

```
('F1', 'F2') 0.0
('DC', 'W2') -200.0
```