Operations Research, Constructor University Bremen, Fall 2023 — Session 2

Prof. Sören Petrat — Sep. 7, 2023

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2. Linear Programming

2.1 Graphical Solutions

We consider examples to illustrate different possibilities for solutions.

1. Similar to introductory example:

- Maximize $z = 3x_1 + 2x_2$ ($\Rightarrow$ line $x_2 = -\frac{3}{2}x_1 + \frac{z}{2}$, i.e., slope $-\frac{3}{2}$)
- with constraints

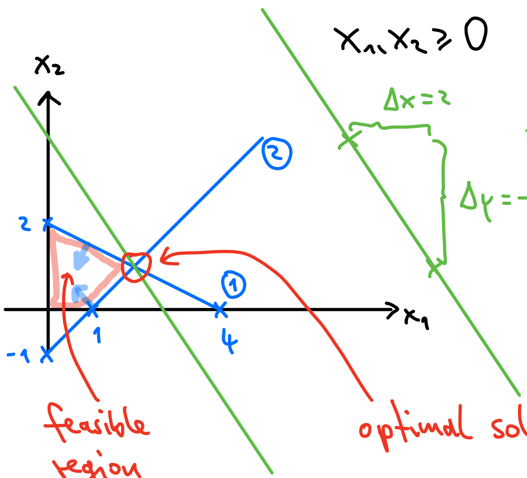
$$x_1 + 2x_2 \le 4 \quad (1)$$

(note: $(1) \Leftrightarrow x_2 \le 2 - \frac{x_1}{2}$, so everything under the line is allowed)

$$x_1 - x_2 \le 1 \quad (2)$$

($(2) \Leftrightarrow x_2 \ge x_1 - 1$, so everything above the line is allowed)

$$x_1, x_2 \ge 0$$



$\Rightarrow$ slope $= \frac{\Delta x_2}{\Delta x_1} = \frac{-3}{2}$ here

feasible region

optimal solution: a "cornerpoint feasible (CPF) solution"

At the optimal corner point (where blue lines meet):

$$x_1 + 2x_2 = 4$$
$$x_1 - x_2 = 1$$

$\Rightarrow$ augmented matrix $\left(\begin{array}{cc|c} 1 & 2 & 4 \\ 1 & -1 & 1 \end{array}\right)$

Gaussian elimination: $R1 - R2 \to R2$ $\left(\begin{array}{cc|c} 1 & 2 & 4 \\ 0 & 3 & 3 \end{array}\right) \xrightarrow{R2/3} \left(\begin{array}{cc|c} 1 & 2 & 4 \\ 0 & 1 & 1 \end{array}\right) \xrightarrow{R1-2R2 \to R1} \left(\begin{array}{cc|c} 1 & 0 & 2 \\ 0 & 1 & 1 \end{array}\right)$

(more next time)

$\Rightarrow$ solution is $\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$, there $z = 3 \cdot 2 + 2 \cdot 1 = 8$.

compare with picture above

$\iff x_2 = -\frac{2}{3}x_1 + \frac{z}{9}$

2. • Minimize $z = 6x_1 + 9x_2$ (slope $-\frac{2}{3}$)

• constraints:

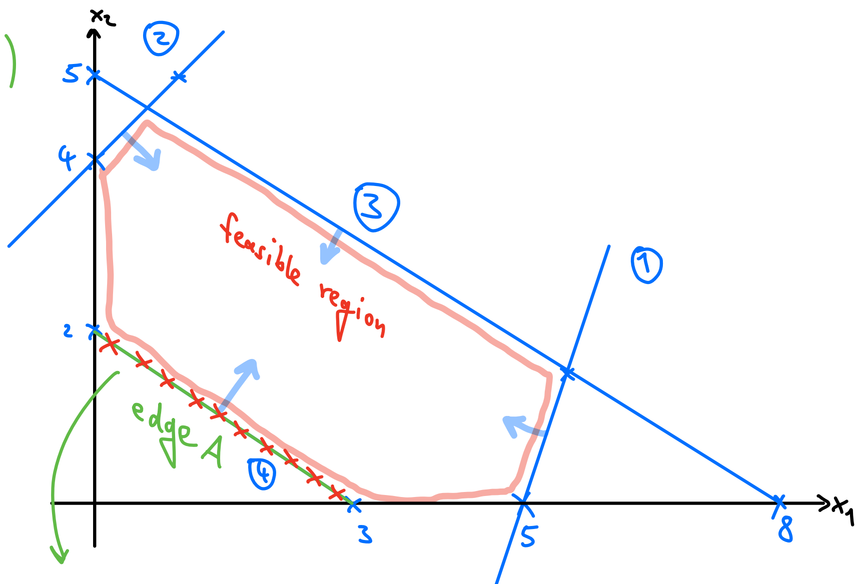
$3x_1 - x_2 \leq 15$ ①

$-x_1 + x_2 \leq 4$ ②

$5x_1 + 8x_2 \leq 40$ ③

$x_2 \geq -\frac{2}{3}x_1 + 2 \iff 2x_1 + 3x_2 \geq 6$ ④

$x_1, x_2 \geq 0$



$z = 18 = 6x_1 + 9x_2$ anywhere on edge A.

Here, the slopes of objective fct. and constraint 4 are the same.

$\Rightarrow$ Any point on edge A is an optimal solution, i.e., there are infinitely many.

meaning infinitely many points on the bounded line segment between points $(0,2)$ and $(3,0)$ (i.e., edge A)

We call such problems "degenerate".

3. • Maximize $z = 4x_1 + 2x_2$

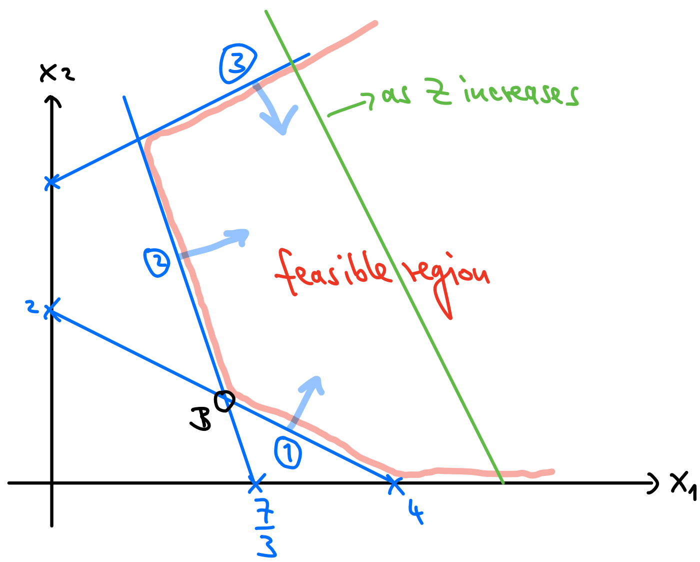
• constraints:

$x_1 + 2x_2 \geq 4$ ①

$3x_1 + x_2 \geq 7$ ②

$-x_1 + 2x_2 \leq 7$ ③

$x_1, x_2 \geq 0$



Here, feasible region is unbounded, and z increases in unbounded direction.

There are (infinitely) many feasible solutions, but none of them is optimal.

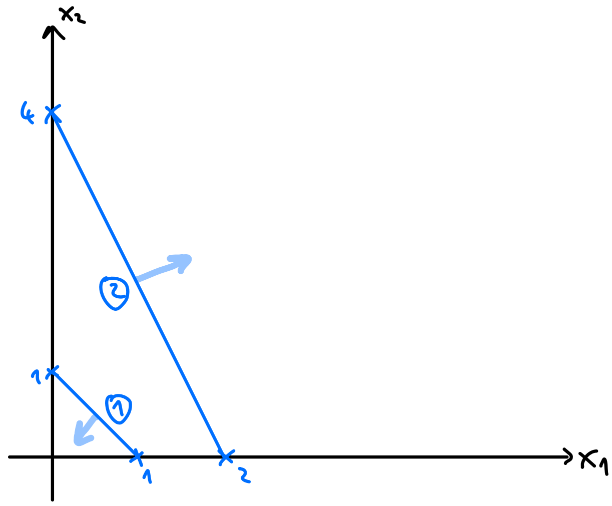
(Note: If z would be minimized, the optimal solution would be at $B = \binom{2}{1}$, and $z = 10$.)

4. • Maximize $z = 3x_1 + 4x_2$

• constraints: $x_1 + x_2 \leq 1$ (1)

$2x_1 + x_2 \geq 4$ (2)

$x_1, x_2 \geq 0$



=> The feasible region is empty; there are no feasible solutions.

We call such problems "over-constrained".

Summary:

The feasible region can be

a) bounded
- either → optimal sol. is CPF only ("Corner Point Feasible")
- or → ∞ many optimal solutions (at least one is a corner point)

b) unbounded
- either → optimal sol. is CPF only
- or → ∞ many optimal solutions
- or → no optimal solution

c) empty → no optimal solution