More generally, two prototypical examples (but mixtures are also possible) of linear Programming (LP) models are:

I) $\boxed{\text{Activity analysis problem:}}$ (e.g., Wyndor)

- A = set of activities (or products)
- R = set of resources (or production facilities)
- w_{ij} = workload required from activity $i \in A$ on resource $j \in R$
- c_j = available capacity of resource $j \in R$
- p_i = profit from performing one unit of activity $i \in A$
- decision variables x_i : # of units of activity $i \in A$ to perform

$$\boxed{\begin{array}{l} \text{LP problem: } \bullet \text{ maximize } Z = \sum_{i \in A} p_i x_i \quad \text{(total profit)} \\ \qquad\qquad\quad \bullet \text{ constraints: } \sum_{i \in A} w_{ij} x_i \le c_j \quad \text{for all } j \in R \text{, and } x_i \geq 0 \text{ for all } i \in A \end{array}}$$

II) $\boxed{\text{Diet-type problem:}}$

- F = set of foods
- N = set of nutrients
- c_i = unit cost of food $i \in F$
- r_j = minimum requirement for nutrient $j \in N$
- a_{ij} = amount of nutrient $j \in N$ from eating one unit of food $i \in F$
- decision variables x_i = # of units of food $i \in F$ to consume

LP problem: • minimize $Z = \sum_{i \in F} c_i x_i$ (total cost)

• constraint: $\sum_{i \in F} a_{ij} x_i \geq r_j$ for all $j \in N$, and $x_i \geq 0$ for all $i \in F$

2.2 Standard Form of LP Problems

Goal: Bring all LP problems into a standardized form. Then later we can easier develop a general algorithm to solve them.

Goal: Write LP problems in the following standard form: (note: some books might use other very similar standards)

• Minimize $Z = c^T x$, with $c \in \mathbb{R}^m$, $x \in \mathbb{R}^m$

• Constraints: $Ax = b$, with A an $n \times m$ matrix, $b \in \mathbb{R}^n$

and $x \geq 0$ (meaning $x_j \geq 0$ for all $j = 1, \dots, m$)

Explanation of notation:

• $c = \begin{pmatrix} c_1 \\ \vdots \\ c_m \end{pmatrix}$, $x = \begin{pmatrix} x_1 \\ \vdots \\ x_m \end{pmatrix}$, $b = \begin{pmatrix} b_1 \\ \vdots \\ b_n \end{pmatrix}$ are column vectors

$c^T = (c_1, \dots, c_m) = c$ transpose $=$ row vector

$\Rightarrow c^T x = (c_1, \dots, c_m) \begin{pmatrix} x_1 \\ \vdots \\ x_m \end{pmatrix} = \sum_{i=1}^{m} c_i x_i$ (= dot product of c and x)

multiplication of a $(1 \times m)$ matrix with an $(m \times 1)$ matrix

• $A = \begin{pmatrix} A_{11} & \cdots & A_{1m} \\ \vdots & \ddots & \vdots \\ A_{n1} & \cdots & A_{nm} \end{pmatrix} = n \times m$ matrix, or $A \in \underbrace{Mat(n,m)}_{\text{set of } n \times m \text{ matrices}}$

Recall: $Ax = \begin{pmatrix} A_{11} & \cdots & A_{1m} \\ \vdots & & \vdots \\ A_{n1} & \cdots & A_{nm} \end{pmatrix} \begin{pmatrix} x_1 \\ \vdots \\ x_m \end{pmatrix} = \begin{pmatrix} A_{11}x_1 + \dots + A_{1m}x_m \\ A_{21}x_1 + \dots + A_{2m}x_m \\ \vdots \\ A_{n1}x_1 + \dots + A_{nm}x_m \end{pmatrix}$

i.e., $(Ax)_i = \sum_{j=1}^{m} A_{ij}x_j$

$\Rightarrow$ $Ax = b$ means: $\sum_{j=1}^{m} A_{ij}x_j = b_i$ for all $i = 1, \dots, n$

Claim: Every LP problem can be written in standard form.

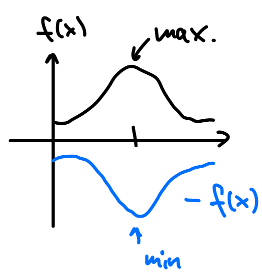
We illustrate this with the following example:

- Maximize $z = x_1 + 2x_2 + 3x_3$

- Constraints:

$$x_1 + x_2 - x_3 = 1 \quad (1)$$
$$-2x_1 + x_2 + 2x_3 \geq -5 \quad (2)$$
$$x_1 - x_2 \leq 4 \quad (3)$$
$$x_2 + x_3 \leq 5 \quad (4)$$
$$x_1 \geq 0 \quad (5)$$
$$x_2 \geq 0 \quad (6)$$

Step 1: Turn maximization into minimization (if necessary).



Our example: Minimize $-z =: \tilde{z} = -x_1 - 2x_2 - 3x_3$

Step 2: Slack variables.

↳ First, write inequalities in standard order: all variables to the left, number to the right, $\leq$ sign.

Our example: Write (2) as $2x_1 - x_2 - 2x_3 \leq 5$.

↳ Then, turn inequalities into equalities + non-negativity constraints by introducing "slack variables" s_i:

Our example: Write 2, 3, 4 as:

$$2x_1 - x_2 - 2x_3 + s_1 = 5 \quad \text{with } s_1 \geq 0,$$
$$x_1 - x_2 + s_2 = 4 \quad \text{with } s_2 \geq 0,$$
$$x_2 + x_3 + s_3 = 5 \quad \text{with } s_3 \geq 0.$$

Step 3: Replace variables without non-negativity constraint by differences.

Our example: x_3 has no nonnegativity constraint

$\Rightarrow$ write $x_3 = u - v$ with $u \geq 0, v \geq 0$.

To summarize, we have rewritten the problem in standard form with:

$$\tilde{x} = \begin{pmatrix} x_1 \\ x_2 \\ u \\ v \\ s_1 \\ s_2 \\ s_3 \end{pmatrix}, \quad A = \begin{array}{c} \begin{array}{ccccccc} x_1 & x_2 & u & v & s_1 & s_2 & s_3 \\ \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow \end{array} \\ \begin{pmatrix} 1 & 1 & -1 & 1 & 0 & 0 & 0 \\ 2 & -1 & -2 & 2 & 1 & 0 & 0 \\ 1 & -1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 1 & -1 & 0 & 0 & 1 \end{pmatrix} \end{array}, \quad b = \begin{pmatrix} 1 \\ 5 \\ 4 \\ 5 \end{pmatrix}, \quad c = \begin{pmatrix} -1 \\ -2 \\ -3 \\ 3 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

$(m = 7, n = 4)$

$$(\tilde{z} = c^T \tilde{x}, \; A\tilde{x} = b, \; \tilde{x} \geq 0).$$

We are now confronted with solving a system of linear eq.s $Ax = b$, with $A \in \mathrm{Mat}(n \times m)$, $b \in \mathbb{R}^n$.

Note:

- As in the example above, for us A is typically a wide matrix $(m > n)$, i.e., the system is underdetermined and there are many solutions.

- In Finite Mathematics you learned about least-norm solutions, i.e., solutions that minimize $\|x\|$. Our goal is: Find solution that optimizes the linear objective function.

Next step: Find all solutions to $Ax = b$ using Gaussian elimination.

(Afterwards: select the optimal one.)