Operations Research, Constructor University Bremen, Fall 2023 — Session 7

Prof. Sören Petrat — Sep. 26, 2023

Last time: We start from the following simplex tableau:

easy choices for initial pivot columns (s_1, s_2, s_3)

$$
\begin{array}{c|ccccccc|c}
 & x_1 & x_2 & u & v & s_1 & s_2 & s_3 & \\
\hline
 & 1 & 1 & -1 & 1 & 0 & 0 & 0 & 1 \\
A & 2 & -1 & -2 & 2 & 1 & 0 & 0 & 5 \\
 & 1 & -1 & 0 & 0 & 0 & 1 & 0 & 4 \quad b \\
 & 0 & 1 & 1 & -1 & 0 & 0 & 1 & 5 \\
\hline
c^T & -1 & -2 & -3 & 3 & 0 & 0 & 0 & 0
\end{array}
$$

want to maximize this entry

choose, e.g., x_1 as extra pivot column

Step (i): Find one basic feasible solution (to initialize the simplex algorithm)

Step (ii): • Entry variable (new pivot column): Column with most negative entry in last row (the c^T-row)

• Leaving variable: Look at ratios of "b-column" to "entry variable column". Choose row with smallest positive ratio.

With this, we last time ended up with:

$$
\begin{array}{ccccccc|c}
x_1 & x_2 & u & v & s_1 & s_2 & s_3 & \\
\hline
1 & -1 & 0 & 0 & 0 & 1 & 0 & 4 \\
0 & -3 & 0 & 0 & 1 & 0 & 0 & 3 \\
0 & -2 & 1 & -1 & 0 & 1 & 0 & 3 \\
0 & 3 & 0 & 0 & 0 & -1 & 1 & 2 \\
\hline
0 & -9 & 0 & 0 & 0 & 4 & 0 & 13
\end{array}
$$

next entry variable: x_2

new pivot (only positive entry in this column) $\Rightarrow$ new leaving variable: s_3

(iii) Repeat: entry variable x_2, leaving variable s_3

$$
\begin{array}{r|ccccccc|c}
 & x_1 & x_2 & u & v & s_1 & s_2 & s_3 & \\
\hline
\frac{1}{3}R_4+R_1 \to R_1: & 1 & 0 & 0 & 0 & 0 & \frac{2}{3} & \frac{1}{3} & \frac{14}{3} \\
R_4+R_2 \to R_2: & 0 & 0 & 0 & 0 & 1 & -1 & 1 & 5 \\
\frac{2}{3}R_4+R_3 \to R_3: & 0 & 0 & 1 & -1 & 0 & \frac{1}{3} & \frac{2}{3} & 13/3 \\
\frac{1}{3}R_4 \to R_4: & 0 & 1 & 0 & 0 & 0 & -\frac{1}{3} & \frac{1}{3} & \frac{2}{3} \\
\hline
3R_4+R_5 \to R_5: & 0 & 0 & 0 & 0 & 0 & 1 & 3 & 19
\end{array}
$$

$\Rightarrow$ $\boxed{x_1 = \frac{14}{3},\ x_2 = \frac{2}{3},\ u = \frac{13}{3},\ s_1 = 5,\ v = 0,\ s_2 = 0,\ s_3 = 0, \text{ and } z = -19}$

all are non-negative

$\Rightarrow$ no further improvement possible

$\Rightarrow$ We found an optimal basic solution.

Another example:

Maximize $z = 2x_1 + x_2$ with constraints

$-x_1 + x_2 \le 1 \quad \Rightarrow \quad -x_1 + x_2 + s_1 = 1$

$x_1 - 2x_2 \le 2 \quad\quad x_1 - 2x_2 + s_2 = 2$

$x_1, x_2 \ge 0$

Simplex tableau:

new entry row

$$
\begin{array}{cccc|c}
x_1 & x_2 & s_1 & s_2 & \\
\hline
-1 & 1 & 1 & 0 & 1 \\
1 & -2 & 0 & 1 & 2 \\
\hline
-2 & -1 & 0 & 0 & 0
\end{array}
$$

least positive ratio (actually the only positive ratio here)

$\Rightarrow s_2$ = leaving variable

both positive, so we can immediately see that a basic feasible solution is $x_1 = 0, x_2 = 0, s_1 = 1, s_2 = 2$ (with $z = 0$).

(We already have a basic feasible solution to start the algorithm.)

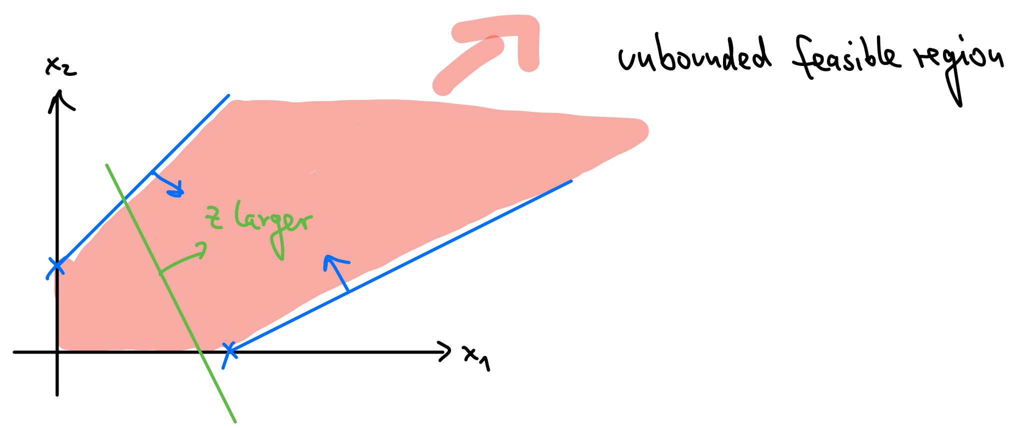
$$
\begin{array}{r|cccc|c}
 & x_1 & x_2 & s_1 & s_2 & \\
\hline
R_2+R_1\to R_1: & 0 & -1 & 1 & 1 & 3 \\
 & 1 & -2 & 0 & 1 & 2 \\
\hline
2R_2+R_3\to R_3: & 0 & -5 & 0 & 2 & 4
\end{array}
$$

$\hookrightarrow x_2$ should be new entry variable

but none of these ratios are positive!

$\Rightarrow$ We can increase x_2 as much as we like (no boundary constraint), i.e., we can make z more negative without bounds.

Graphically:



Summary of simplex algorithm:

- Step (i): Find a basic feasible solution.

  Note: sometimes it is not easy to choose a feasible basic solution to start with; we will deal with that later. If no basic feasible solution can be found, the feasible region is empty.

- Step (ii): Find entering and leaving variables and perform Gaussian elimination.
  - Entering variable column: choose the one with most negative entry in objection function row. If all entries are non-negative, solution has been found (terminate the algorithm).

    (If more than one value is the most negative, we can choose one of them at random.)
  - Leaving variable: choose row with the least positive ratio of right-hand coefficient to coefficient in that column.

    (If more than one ratio is the least positive, we can choose one of them at random.)

    If not possible (if all coefficients in column are negative), we have found that the feasible region is unbounded and objective function can be made arbitrarily small.

- Repeat step (ii) if necessary.

Note: The simplex algorithm only finds one optimal solution, even though there might be (infinitely) many.