Operations Research, Constructor University Bremen, Fall 2023 — Session 8

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## 2.4 The Dual LP Problem

General goal: Write the objective fct. in terms of the capacities b. This could tell us about how the profit changes when the capacities change.

We consider another example:

A factory produces cars ($x_1$) and trucks ($x_2$). We aim at maximizing the profit $z = \underbrace{3}_{\text{profit per car}}x_1 + \underbrace{2}_{\text{profit per truck}}x_2$, subject to the constraints

$\longrightarrow$ minimize $z' = -3x_1 - 2x_2$

$5x_1 \le 100$ (car assembly; need 5 h per car; 100 h available) (1)

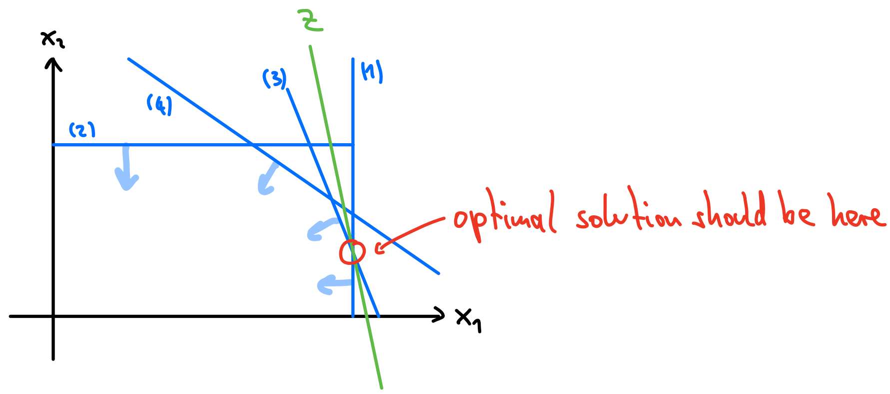
$10x_2 \le 100$ (truck assembly) (2)

$4x_1 + 3x_2 \le 100$ (metal stamping) (3)

$3x_1 + 5x_2 \le 100$ (engine assembly) (4)

$x_1, x_2 \ge 0$

Graphically (rough sketch):



We start by computing the optimal solution via the simplex method:

we already have a basic feasible solution to start from

| | $x_1$ | $x_2$ | $s_1$ | $s_2$ | $s_3$ | $s_4$ | |
|---|---|---|---|---|---|---|---|
| | 5 | 0 | 1 | 0 | 0 | 0 | 100 |
| | 0 | 10 | 0 | 1 | 0 | 0 | 100 |
| | 4 | 3 | 0 | 0 | 1 | 0 | 100 |
| | 3 | 5 | 0 | 0 | 0 | 1 | 100 |
| | -3 | -2 | 0 | 0 | 0 | 0 | 0 |

entry variable: $x_1$

leaving variable: $s_1$

| | $x_1$ | $x_2$ | $s_1$ | $s_2$ | $s_3$ | $s_4$ | |
|---|---|---|---|---|---|---|---|
| $R_1/5 \to R_1$: | 1 | 0 | $\frac{1}{5}$ | 0 | 0 | 0 | 20 |
| | 0 | 10 | 0 | 1 | 0 | 0 | 100 |
| $-\frac{4}{5}R_1 + R_3 \to R_3$: | 0 | 3 | $-\frac{4}{5}$ | 0 | 1 | 0 | 20 |
| $-\frac{3}{5}R_1 + R_4 \to R_4$: | 0 | 5 | $-\frac{3}{5}$ | 0 | 0 | 1 | 40 |
| $\frac{3}{5}R_1 + R_5 \to R_5$: | 0 | -2 | $\frac{2}{5}$ | 0 | 0 | 0 | 60 |

entry variable: $x_2$

leaving variable: $s_3$

| | $x_1$ | $x_2$ | $s_1$ | $s_2$ | $s_3$ | $s_4$ | |
|---|---|---|---|---|---|---|---|
| | 1 | 0 | $\frac{1}{5}$ | 0 | 0 | 0 | 20 |
| $-\frac{10}{3}R_3 + R_2 \to R_2$: | 0 | 0 | $\frac{40}{15}$ | 1 | $-\frac{10}{3}$ | 0 | $\frac{100}{3}$ |
| $R_3/3 \to R_3$: | 0 | 1 | $-\frac{4}{15}$ | 0 | $\frac{1}{3}$ | 0 | $\frac{20}{3}$ |
| $-\frac{5}{3}R_3 + R_4 \to R_4$: | 0 | 0 | $\frac{11}{15}$ | 0 | $-\frac{5}{3}$ | 1 | $\frac{20}{3}$ |
| $\frac{2}{3}R_3 + R_5 \to R_5$: | 0 | 0 | $\frac{1}{15}$ | 0 | $\frac{2}{3}$ | 0 | $\frac{220}{3}$ |

Done!

=> Optimal solution: $x_1 = 20$, $x_2 = \frac{20}{3}$, $s_2 = \frac{100}{3}$, $s_4 = \frac{20}{3}$, $s_1 = 0$, $s_3 = 0$, with $z' = -\frac{220}{3}$

(meaning profit $z = -z' = \frac{220}{3}$)

Now note:

- Constraints (1) and (3) hold with equality (no slack: $s_1 = 0$, $s_3 = 0$). These are the binding constraints.
- Constraints (2) and (4) are non-binding.

Knowing the binding constraints, we can write our solution in the following way:

(1) and (3) are the binding constraints

Let $\tilde{A} = \begin{array}{cl} \begin{array}{cc} x_1 & x_2 \end{array} & \\ \begin{pmatrix} 5 & 0 \\ 4 & 3 \end{pmatrix} & \begin{array}{l} \leftarrow \text{constraint (1)} \\ \leftarrow \text{constraint (3)} \end{array} \end{array}$ $\quad \tilde{b} = \begin{pmatrix} 100 \\ 100 \end{pmatrix}$, want $\tilde{A}x = \tilde{b}$.

Since $\tilde{A}$ is now a square $(n \times n)$ matrix with full rank, it can be inverted: $x = \tilde{A}^{-1}\tilde{b}$.

$$\Rightarrow z = \underbrace{c}_{\substack{=(3,2) \\ \text{in our example}}}{}^{T} x = \underbrace{c^T \tilde{A}^{-1}}_{=:\tilde{y}^T} \tilde{b} = y^T b \quad \text{, where } \tilde{y} \in \mathbb{R}^2,$$

$$b = \begin{pmatrix} b_1 \\ b_2 \\ b_3 \\ b_4 \end{pmatrix} = \begin{pmatrix} 100 \\ 100 \\ 100 \\ 100 \end{pmatrix}$$

and where we defined $y := \begin{pmatrix} \tilde{y}_1 \\ 0 \\ \tilde{y}_2 \\ 0 \end{pmatrix}$. Putting 0 at the non-binding constraints; here: (2) and (4)

Note: For $2\times 2$ matrices, the formula for the inverse is: $\begin{pmatrix} a & b \\ c & d \end{pmatrix}^{-1} = \frac{1}{ad-bc}\begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$.

In our example: $\tilde{y}^T = c^T \tilde{A}^{-1} = (3,2)\begin{pmatrix} 5 & 0 \\ 4 & 3 \end{pmatrix}^{-1} = (3,2)\,\frac{1}{15}\begin{pmatrix} 3 & 0 \\ -4 & 5 \end{pmatrix}$

$$= \frac{1}{15}(1, 10).$$

$$\Rightarrow y = \frac{1}{15}\begin{pmatrix} 1 \\ 0 \\ 10 \\ 0 \end{pmatrix}$$

Test: $\tilde{y}^T \tilde{b} = \frac{1}{15}(1, 10)\begin{pmatrix} 100 \\ 100 \end{pmatrix} = \frac{1100}{15} = \frac{220}{3} \quad (= z) \quad \checkmark$

Test: $x = \tilde{A}^{-1} b = \frac{1}{15}\begin{pmatrix} 3 & 0 \\ -4 & 5 \end{pmatrix}\begin{pmatrix} 100 \\ 100 \end{pmatrix} = \begin{pmatrix} \frac{300}{15} \\ \frac{100}{15} \end{pmatrix} = \begin{pmatrix} 20 \\ 20/3 \end{pmatrix} \quad \checkmark$

Next time: What if capacities are changed by a small amount?