Operations Research, Constructor University Bremen, Fall 2023 — Session 9

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Last time: We started from the LP problem
- maximize $z = c^T x$
- constraints $Ax \le b$, $x \ge 0$.

Knowing about the optimal solution, we found that the obj. fct. can be written as

$$z = b^T y, \text{ where in our example } y = \frac{1}{15}\begin{pmatrix}1\\0\\10\\0\end{pmatrix}.$$

Now: Change capacities b by a small amount; small meaning the binding constraints remain the same.

Then $b \longrightarrow b + \delta$ and we find $x = \tilde{A}^{-1}(\tilde{b} + \tilde{\delta})$.

$$\Rightarrow \text{new profit } z(\delta) = y^T(b+\delta) = \underbrace{y^T b}_{= z(0) = \text{current profit}} + \overbrace{y^T \delta}^{= \text{extra profit/loss from changed capacities}}$$

The $y_1, \dots, y_m$ are called $\boxed{\text{shadow prices.}}$ These are the changes of profit per unit of capacity at current operating conditions.

In our example, we found $y = \begin{pmatrix}1/15\\0\\\frac{10}{15}\\0\end{pmatrix}$. So for example, increasing the capacity b_1 of constraint (1) by one unit will increase the profit by $\frac{1}{15}$.

If we could choose to increase the working hours for either constraint, we should choose constraint (3) because this increases the profit the most.

Increasing or decreasing b_2 or b_4 by a small amount will not change the profit since $y_2 = 0 = y_4$.

Next: How to compute shadow prices directly (via solving the "dual" LP problem).

Recall the example: maximize profit $z = 3x_1 + 2x_2 = c^T x$

with constraints

$$\left.\begin{aligned} 5x_1 &\leq 100 \\ 10x_2 &\leq 100 \\ 4x_1 + 3x_2 &\leq 100 \\ 3x_1 + 5x_2 &\leq 100 \end{aligned}\right\} Ax \leq b$$

$$x_1, x_2 \geq 0$$

Now: Consider the following scenario: A company wants to buy our production capacity.

What are fair prices y_1, y_2, y_3, y_4 for the resources (1), (2), (3), (4)?

In our example:
- profit per car: 3
- profit per truck: 2
- current car assembly hours: 5 for constraint (1), 4 for constraint (3), 3 for constraint (4)
- trucks: 10 for (2), 3 for (3), 5 for (4)

Thus we want:

$$\left.\begin{aligned} &\cdot\ 5y_1 + 4y_3 + 3y_4 \geq 3 \\ &\cdot\ \underbrace{10y_2 + 3y_3 + 5y_4}_{=A^T y} \geq \underbrace{2}_{c} \end{aligned}\right]$$ selling capacity to produce one car/truck needs to be at least as profitable as producing a car/truck

(Recall: $A = \begin{pmatrix} 5 & 0 \\ 0 & 10 \\ 4 & 3 \\ 3 & 5 \end{pmatrix} \Rightarrow A^T = \begin{pmatrix} 5 & 0 & 4 & 3 \\ 0 & 10 & 3 & 5 \end{pmatrix}$ is the transpose of A.)

The price for all capacity is $y_1 \cdot 100 + \ldots + y_4 \cdot 100 = b^T y$. Minimizing this yields the minimum prize = fair price for the production capacity.

This leads to the "dual problem":
- minimize $b^T y$,
- subject to $A^T y \geq c$ and $y \geq 0$.

as compared to the original "primal problem":
- maximize $c^T x$,
- subject to $Ax \leq b$ and $x \geq 0$.

Solving the dual problem gives us the shadow prices.

Two results about the relation between dual and primal LP:

- Note that $c^T x = x^T c \underset{c \leq A^T y}{\leq} x^T A^T y \underset{(AB)^T = B^T A^T}{=} (Ax)^T y \underset{Ax \leq b}{\leq} b^T y$.

  This is known as weak duality:

  If x is a solution to the primal problem (i.e., x is feasible, but not necessarily optimal), and y is a solution to the dual problem, then $c^T x \leq b^T y$.

- A bit harder to prove (but intuitively clear) is strong duality:

  The dual has an optimal solution if and only if the primal does. In this case $c^T x = b^T y$.