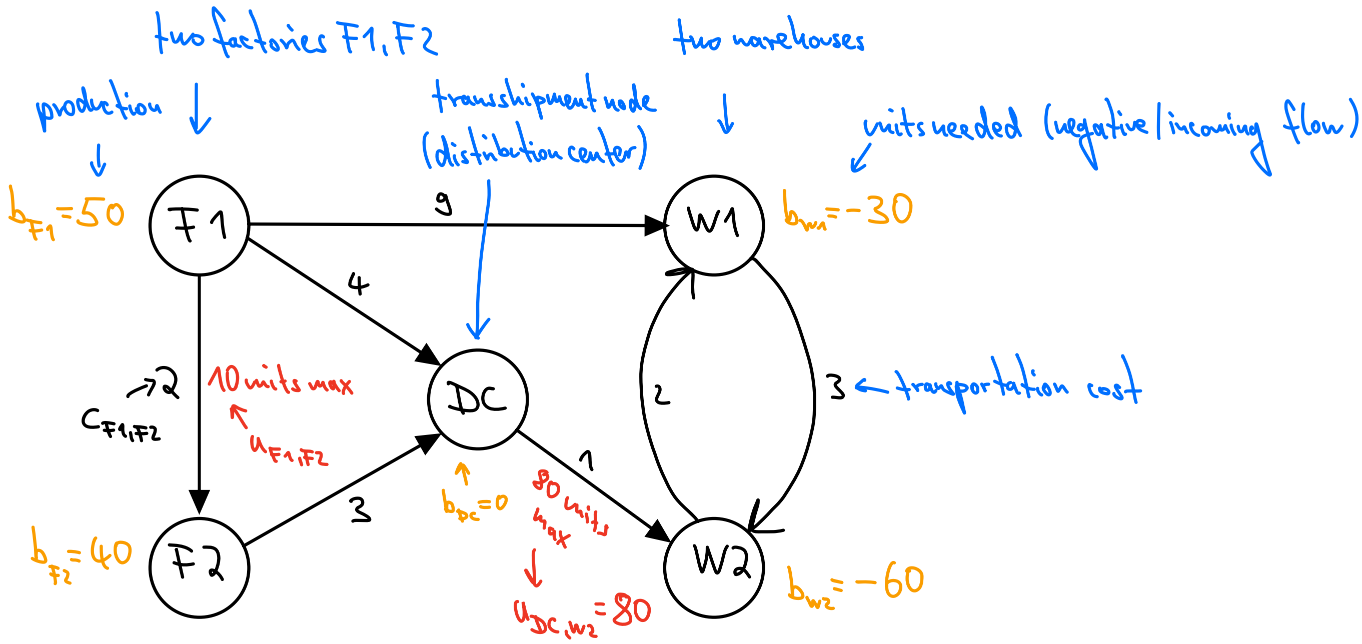
Operations Research, Constructor University Bremen, Fall 2023 — Session 14

Prof. Sören Petrat — Oct. 24, 2023

Today, we discuss $\boxed{\text{minimum cost flow problems.}}$

Example (Hillier, Lieberman Chapters 3.4 and 9.6):



Set of nodes $N=\{F1,F2,DC,W1,W2\}$

Set of arcs $A=\{(F1,F2),(F1,DC),(F1,W1),(F2,DC),(DC,W2),(W1,W2),(W2,W1)\}$

General formulation:
- nodes $i\in N$
- directed arcs $(i,j)\in A$
- c_{ij}: unit cost of transportation on arc (i,j)
- u_{ij}: max. capacity on arc (i,j)
- node constraints
  - $b_i>0$ for supply/source nodes
  - $b_i<0$ for demand/sink nodes
  - $b_i=0$ for transshipment nodes
- x_{ij}: flow from i to j (decision variables)

LP formulation: • Minimize cost $z = \sum_{(i,j)\in A} c_{ij} x_{ij}$

• Constraints: $\underbrace{\sum_j x_{ij}}_{\text{outgoing flow at node } i} - \underbrace{\sum_j x_{ji}}_{\text{incoming flow at node } i} = b_i$ for all nodes $i \in N$

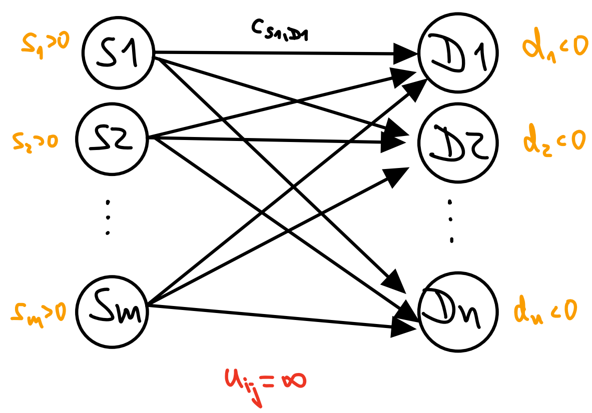
and $0 \le x_{ij} \le u_{ij}$ for all arcs $(i,j) \in A$.

Note: Similarly as discussed before:

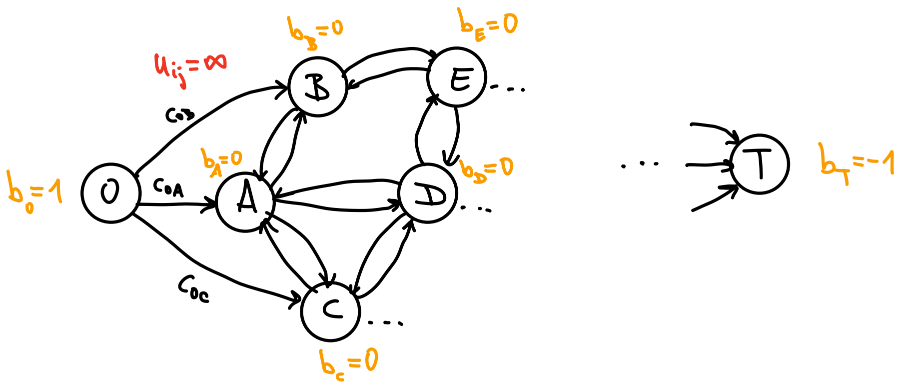
- One can show that a necessary condition for feasible solutions is $\sum_i b_i = 0$ (supply = demand). This can always be achieved by introducing dummy nodes (similarly as we discussed before).
- All basic variables in all basic feasible solutions are integer, if all b_i and u_{ij} are integer.
- A faster network simplex method is available.

How can our previous cases be formulated as min. cost flow problems?

- Transportation problem: – only supply and demand nodes (no transshipment nodes), all supply nodes connected to all demand nodes
  – all $u_{ij} = \infty$ since no upper bound constraints



- Shortest Path problem:
  - origin = supply node with $b_0 = 1$
  - destination = demand node with $b_T = -1$
  - other nodes are transshipment, i.e., $b_i = 0$.
  - draw all arcs in both directions (except source/sink)
  - all $u_{ij} = \infty$
  - c_{ij} = distances as given (so min. cost = min. distance)



- Max Flow problem:
  - all $c_{ij} = 0$
  - source $b_0 = F$ $\underbrace{\text{large}}$ (larger than a good guess for the max. flow given the u_{ij}), sink $b_T = -F$, all other nodes $b_i = 0$
  - u_{ij} as given
  - extra arc from source to sink with $c_{0T} = M$ very large (and $u_{0T} = \infty$)
    - ↳ so supply-demand constraints can be satisfied (solutions exist)
    - ↳ then $c_{ij} = 0$ arcs are preferred, rest is sent through c_{0T} arc at high cost