Operations Research, Constructor University Bremen, Fall 2023 Session 16

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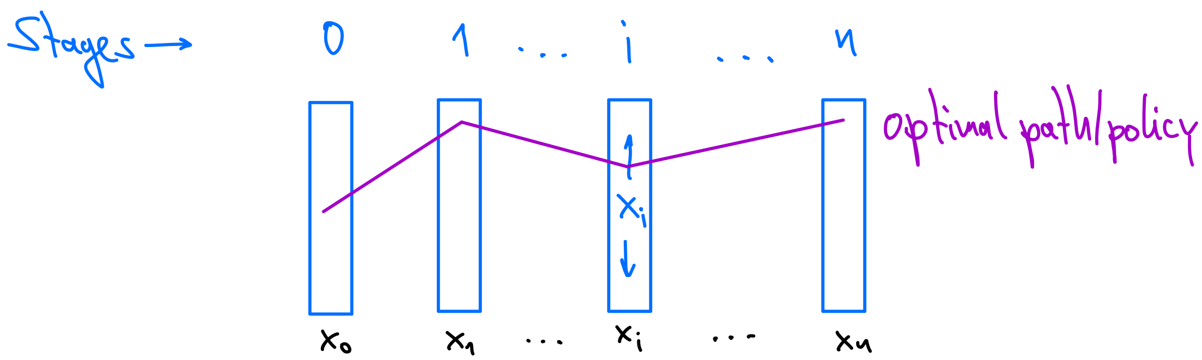
3. Further Optimization Techniques

3.1 Dynamic Programming

Setting of Dynamic Programming:

- Often, a problem can be divided into simpler subproblems or problem "stages"
- A "policy decision" (= transition from one "state" to another) needs to be made at each stage
- Goal: find optimal sequence of decisions (= "optimal policy"), i.e., solutions to the subproblems, in a recursive way

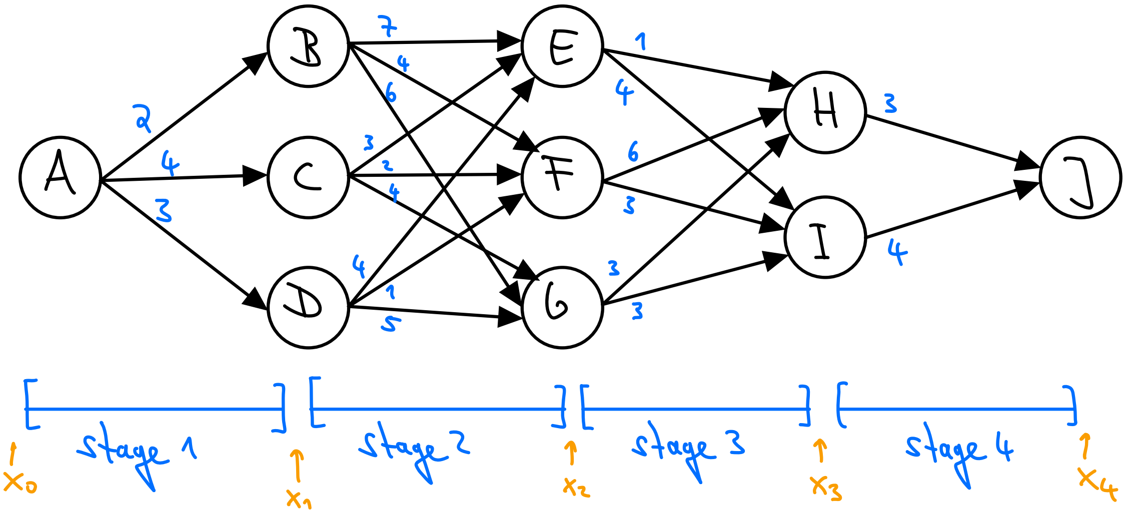


Decision variables: $x_i = \underbrace{\text{state to transition to in stage } i}_{= \text{ possible transitions at stage } i}$ (from some state at stage $i-1$).

Example: Stagecoach problem (Hillier, Lieberman: Chapter 10.1)

Need to travel from A to J; travel/insurance costs are associated to different route segments.

This is just a special type of shortest path problem: one with a natural notions of stages.

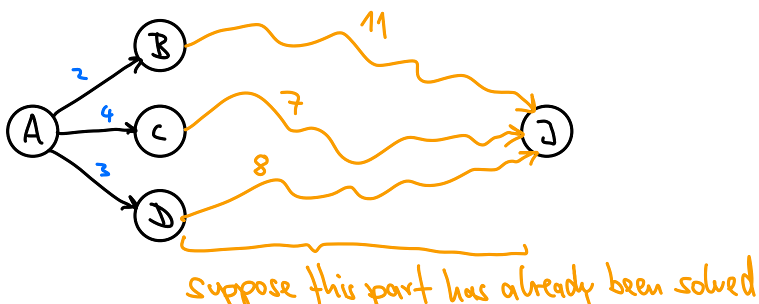


Route: $A = x_0 \longrightarrow x_1 \longrightarrow x_2 \longrightarrow x_3 \longrightarrow x_4 = J$

We know how to solve this with the shortest path algorithm, but here we want to introduce a more general solution procedure using the "subproblem structure".

Dynamic Programming solution idea:

Recursive solution: E.g.:



$\Rightarrow$ two shortest paths: $A \to C \rightsquigarrow J$

$A \to D \underbrace{\rightsquigarrow J}_{\text{optimal rest of path}}$

Recursive procedure:

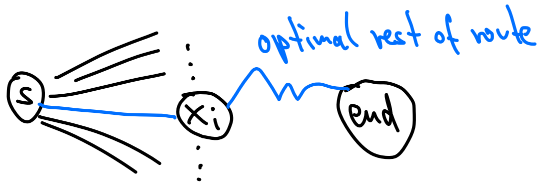
- Start at end (at the goal) (or at the beginning, depending on the problem setting)
- Solve problem: optimal path from stage i-1 to stage i

  $\Rightarrow$ this gives us optimal path from i-1 till end
- Repeat until start is reached

More concretely, we introduce:

- $f_i(s, x_i)$ = cost of travel route starting at s at stage $i-1$, passing through x_i at stage i, and then optimal till the end

  = optimal route from s through x_i till end



- $f_i^*(s) = \min\limits_{x_i} f_i(s, x_i)$ = cost of optimal travel route starting at s (going till end)

  Let x_i^* denote the minimum (not necessarily unique) = optimal choice at stage i

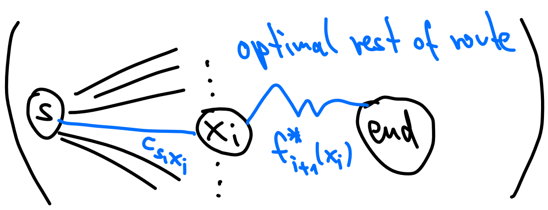
Here we solve an optimization problem

In our example:

$$f_i(s, x_i) = \underbrace{c_{s, x_i}}_{} + \underbrace{f_{i+1}^*(x_i)}_{}$$

cost of travel from s to x_i; see network above (given parameters)

optimal future cost to travel from x_i to the end



Solution procedure for our example:

stage $i = 4$: $(x_4 = J,\ s \in \{H, I\})$

s	f_4^*	x_4^*
H	3	J
I	4	J

$i = 3$:

s	$f_3(s,x_3) = c_{s,x_3} + f_4^*(x_3)$: $x_3 = H$	$x_3 = I$	$f_3^*(s)$	x_3^*
E	$\underbrace{1}_{c_{E,H}} + \underbrace{3}_{f_4^*(H)} = 4$	$4 + 4 = 8$	4	H
F	$6 + 3 = 9$	$3 + 4 = 7$	7	I
G	$3 + 3 = 6$	$3 + 4 = 7$	6	H

$i = 2$

s	$f_2(s,x_2) = c_{s,x_2} + f_3^*(x_2)$ $x_2 = E$	$x_2 = F$	$x_2 = G$	$f_2^*(s)$	x_2^*
B	$7+4=11$	$4+7=11$	$6+6=12$	11	E or F
C	$3+4=7$	$2+7=9$	$4+6=10$	7	E
D	$4+4=8$	$1+7=8$	$5+6=11$	8	E or F

$i = 1$

s	$f_1(s,x_1) = c_{s,x_1} + f_2^*(x_1)$ $x_1 = B$	$x_1 = C$	$x_1 = D$	$f_1^*(s)$	x_1^*
A	$2+11=13$	$4+7=11$	$3+8=11$	11	C or D

$\Rightarrow$ Minimal cost is 11, optimal paths are

A-C-E-H-J or A-D-E-H-J or A-D-F-I-J

(Note: here, the problem is symmetric, so we could have also started at A and solve recursively till we reach J.)