The local job shop problem continued.

Recall:

Season	Spring	Summer	Autumn	Winter	Spring	...
Requirement	255	220	240	200	255	...

$$f_n(s_n, x_n) = \underbrace{200(x_n - s_n)^2}_{\text{employment change cost}} + \underbrace{2000(x_n - r_n)}_{\text{extra employment cost}} + \underbrace{f^*_{n+1}(x_n)}_{\text{optimal costs for later stages}}$$

} cost given employment s_n at stage $n-1$, and x_n at stage n, and optimal future cost

$$f^*_n(s_n) = \min_{r_n \le x_n \le 255} f_n(s_n, x_n)$$

} optimal cost given employment s_n at stage $n-1$, and optimal in the future

Solution:

Stage $n=4$:

s_4	$f^*_4(s_4)$	x^*_4
$200 \le s_4 \le 255$	$200(255-s_4)^2$	255

Stage $n=3$: (winter to spring)

$$f^*_3(s_3) = \min_{200 \le x_3 \le 255} f_3(s_3, x_3)$$

$$= \min_{200 \le x_3 \le 255} \Big[200(x_3 - s_3)^2 + 2000(x_3 - 200) + \underbrace{200(255 - x_3)^2}_{f^*_4(x_3)} \Big]$$

We find the minimum by setting the partial derivative w.r.t. x_3 to zero (keeping s_3 fixed):

$$\frac{\partial f_3(s_3, x_3)}{\partial x_3} = 400(x_3 - s_3) + 2000 - 400(255 - x_3) = 400(2x_3 - s_3 - 250) \stackrel{!}{=} 0$$

$\Rightarrow x_3^* = \frac{250+s_3}{2}$

Since $\frac{\partial^2 f_3(s_3,x_3)}{\partial x_3^2} = 800 > 0$, we indeed have a minimum.

Possible s_3: $240 \le s_3 \le 255 \Rightarrow 245 \le x_3^* \le 252.5$ is feasible $(200 \le x_3 \le 255)$, so x_3^* is always the feasible minimum.

$$\Rightarrow f_3^*(s_3) = 200\left(x_3^* - s_3\right)^2 + 2000\left(x_3^* - 200\right) + 200\left(255 - x_3^*\right)^2$$

$$= 200\underbrace{\left(\frac{250+s_3}{2} - s_3\right)^2}_{= \left(\frac{250-s_3}{2}\right)^2 = \frac{1}{4}(250-s_3)^2} + 2000\underbrace{\left(\frac{250+s_3}{2} - 200\right)}_{= \frac{-150+s_3}{2}} + 200\underbrace{\left(255 - \left(\frac{250+s_3}{2}\right)\right)^2}_{= \left(\frac{260-s_3}{2}\right)^2}$$

$\Rightarrow$

s_3	$f_3^*(s_3)$	x_3^*
$240 \le s_3 \le 255$	$50(250-s_3)^2 + 50(260-s_3)^2 + 1000(s_3-150)$	$\frac{250+s_3}{2}$

One proceeds similarly with stages 2 and 1, but we skip the details.

The results are:

$n=2$:

s_2	$f_2^*(s_2)$	x_2^*
$220 \le s_2 \le 240$	$200(240-s_2)^2 + 115.000$	240
$240 \le s_2 \le 255$	$\frac{200}{9}\left[(240-s_2)^2 + (255-s_2)^2 + (270-s_2)^2\right] + 2000(s_2-195)$	$\frac{2s_2+240}{3}$

$n=1$:

s_1	$f_1^*(s_1)$	x_1^*
255	185.000	247.5

$\Rightarrow x_1^* = 247.5$, $x_2^* = \underbrace{245}_{= \frac{2 \cdot 247.5 + 240}{3}}$, $x_3^* = \underbrace{247.5}_{= \frac{250+245}{2}}$, $x_4^* = 255$, and minimum cost is 185.000 \$.

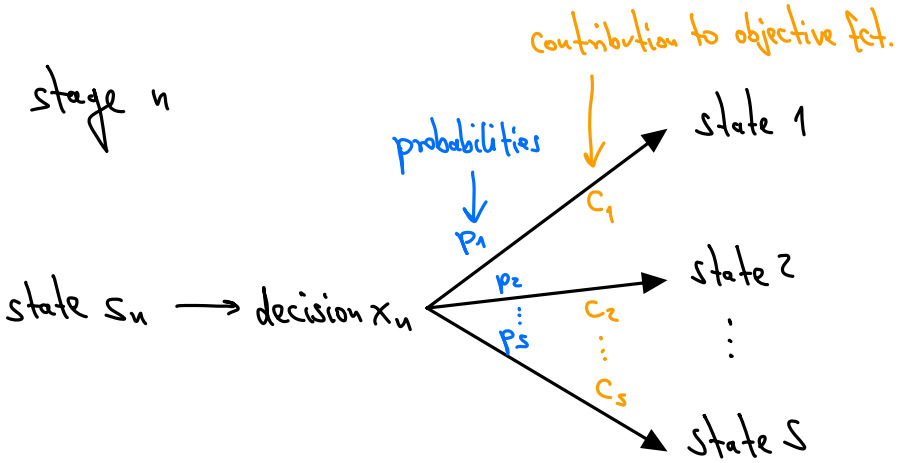
To summarize:

Season	Spring	Summer	Autumn	Winter	Spring	...
Requirement	255	220	240	200	255	...
optimal solution	255	247.5	245	247.5	255	...

Next: Optimization problems which involve probability

We start with Probabilistic Dynamic Programming.

Basic structure:



Objective: usually minimize the expected sum of contributions, e.g., costs.

Example: Hit and Miss Manufacturing Co. (Hillier, Lieberman Chapter 10.4)

Setup: • product produced meets strict quality requirements only with probability $p = \frac{1}{2}$

=> if x items are produced, the probability for producing only bad items is $\left(\frac{1}{2}\right)^x$ (probability that at least one is good is $1-\left(\frac{1}{2}\right)^x$) (binomial distribution)

(note: extra produced items are called "reject allowance")

• for each new batch there are: • 300 $ setup costs
• 100 $ cost per item

- at most 3 batches can be started, items can be inspected after each batch
- if no good item produced, there is a penalty of 1600 $
- objective: choose production schedule to minimize costs
- decision variables: x_n = # of items to produce in batch/stage $n = 1, 2, 3$
- state s = # of acceptable items that still need to be produced = 0 or 1.

done, have produced a good one

no good item yet, might need to continue with next batch