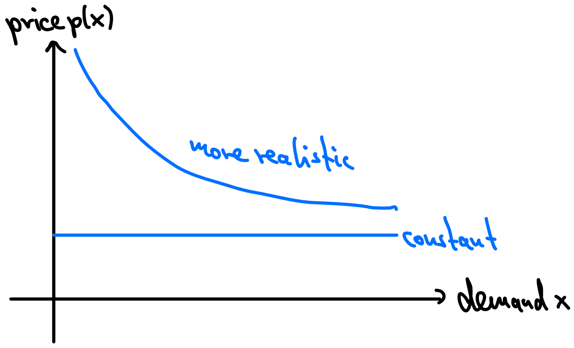
## 3.4 Nonlinear Programming

A general nonlinear programming problem has the standard form:

Maximize $f(x)$ subject to $g_i(x) \leq b_i$ for $i = 1, \dots, m$, and $x \geq 0$, where $x = (x_1, \dots, x_n)$.

A few examples how non-linear $f$ or $g_i$ can arise:

- Wyndor Glass Co. (product mix problem): unit profit often not fixed, but there is a price-demand curve, e.g.,




Then profit $Z(x) = x\,p(x) - \underbrace{c}_{\text{unit production cost: sometimes also a function of } x} x$

Next, let us look at specific examples that highlight the differences to linear Programming.

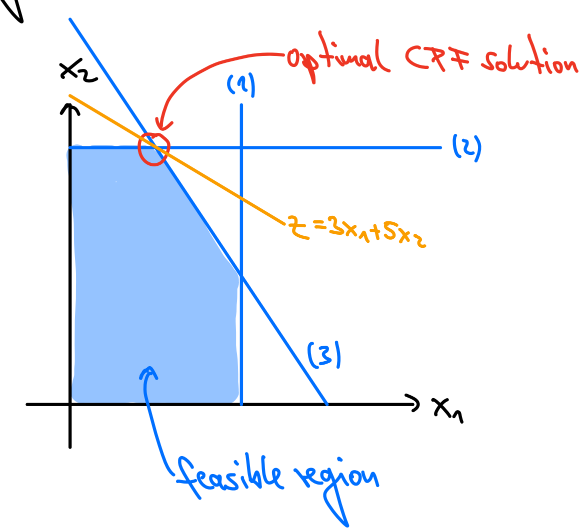
Let us use Wyndor Glass Co. as underlying example. Recall the LP version:

- Maximize $Z = 3x_1 + 5x_2$,
- subject to $x_1 \leq 4$ (1)

  $2x_2 \leq 12$ (2)

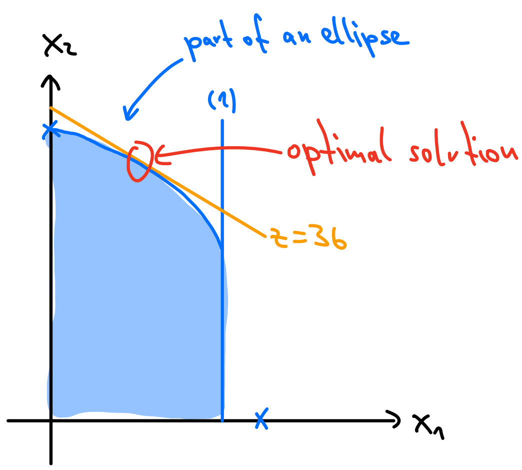
  $3x_1 + 2x_2 \leq 18$ (3)

  $x_1, x_2 \geq 0$




Nonlinear variations:

(1) Nonlinear constraint: replace (2) and (3) by $9x_1^2 + 5x_2^2 \leq 216$




Optimal solution is $(2,6)$:

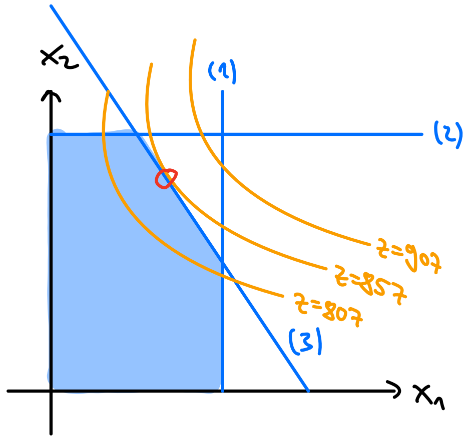
↳ at the boundary of feasible region

↳ but not a cornerpoint anymore

=> simplex method (which goes through CPF solutions only) does not work anymore

(2) Nonlinear objective function, linear constraints:

Maximize $Z = 126x_1 - 9x_1^2 + 182x_2 - 13x_2^2$ subject to (1), (2), (3).




Optimal solution is at boundary of feasible region, but not at a cornerpoint.

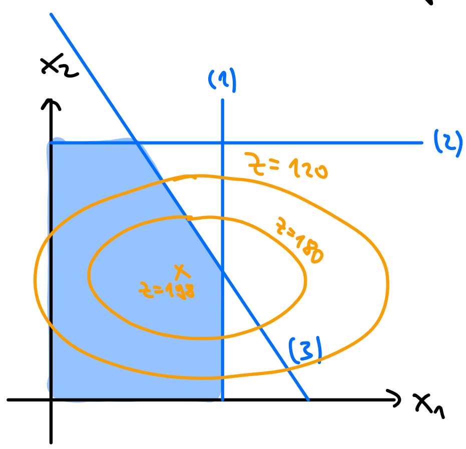
(3) Maximize $Z = 54x_1 - 9x_1^2 + 78x_2 - 13x_2^2$ subject to (1), (2), (3).

Let us check what the maximum of $Z$ without constraints is:

$$\frac{\partial Z}{\partial x_1} = 54 - 18x_1 \overset{!}{=} 0 \implies x_1 = 3$$

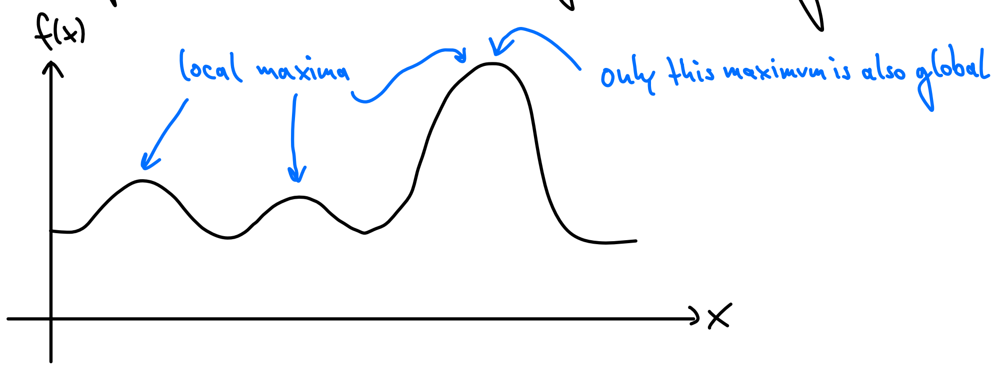
$$\frac{\partial Z}{\partial x_2} = 78 - 26x_2 \overset{!}{=} 0 \implies x_2 = 3$$

Since $Z(x_1, x_2)$ is a downward open parabola, there is only one maximum $(x_1, x_2) = (3, 3)$. This is within the feasible region. Here, $Z = 198$.




$\Rightarrow$ Optimal solution might be inside feasible region!

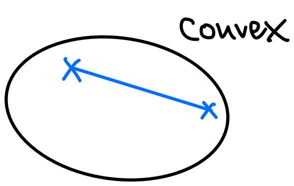
Another problem: A local maximum might not be a global maximum

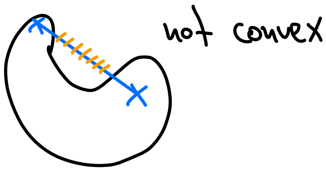



Nonlinear Programming algorithms are generally unable to distinguish between local and global maxima, so they might get stuck in a local maximum (and not find the global one).

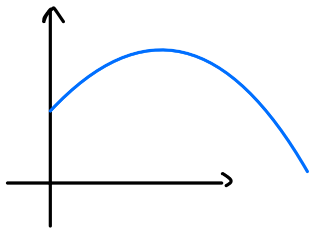
But there are reasonable sufficient conditions for a maximum to be global:

(i) The feasible region is convex, i.e., for any two points in the feasible region, the line segment between them is also in the feasible region.

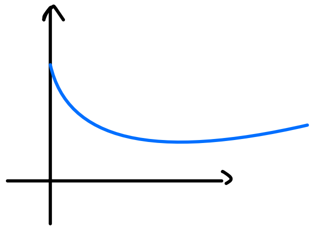



(ii) For maxima: objective function is always "curving downward", i.e., concave:



For minima: objective function is always "curving upward", i.e., convex:



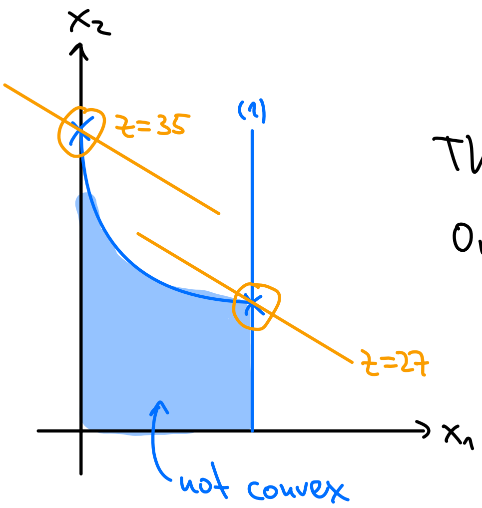
For such convex optimization problems, solvers such as ipopt work well.

(4) Feasible region not convex: • Maximize $Z = 3x_1 + 5x_2$,

• subject to $x_1 \leq 4, \quad x_2 \leq 7$

$$8x_1 - x_1^2 + 14x_2 - x_2^2 \leq 49,$$

$$x_1, x_2 \geq 0.$$




There are two local maxima: (4,3) and (0,7).

Only (0,7) is global, but algorithm might get stuck at (4,3).