

# Advanced Calculus and Methods of Mathematical Physics

## Homework 2

Due on February 21, 2023, before class!

### Problem 1 [5 points]

Give a proof of the (convergence part of the) ratio test that was discussed in class, i.e., prove that  $\sum_{k=0}^{\infty} a_k$  converges if  $\lim_{k \rightarrow \infty} |a_{k+1}/a_k| < 1$ . *Hint: Suppose  $|a_{k+1}/a_k|$  converges to  $r < 1$ . Then, for large enough  $N$ ,  $|a_{N+1}/a_N| < R$  for some other  $r < R < 1$ . What can you conclude, using your knowledge about the geometric series?*

### Problem 2 [4 points]

Use any of the convergence tests discussed in class and in the lecture notes to determine whether the following series converges:

(a)  $\sum_{k=0}^{\infty} (-1)^k \frac{1}{k+1}$ ,

(b)  $\sum_{k=0}^{\infty} \frac{1}{k+1}$  (*Hint: use the comparison or integral test*),

(c)  $\sum_{k=1}^{\infty} \frac{1}{k^2}$ ,

(d)  $\sum_{k=1}^{\infty} \frac{e^k}{k}$ .

### Problem 3 [2 points]

Determine the radius of convergence  $\rho$  of the following power series:

(a)  $\sum_{k=0}^{\infty} x^k$ ,

(b)  $\sum_{k=1}^{\infty} (-1)^k \frac{x^{2k}}{(2k)!}$ .

### Problem 4 [5 points]

Compute the Taylor series of  $f(x) = \ln(1+x)$  around  $c = 0$ , for  $-1 < x < 1$ . Show that the rest term indeed converges to 0 (for any  $-1 < x < 1$ ). Does the Taylor series also converge for  $x = 1$ ? Does it converge for  $x = -1$ ? *Hint: Consider the integral form of the remainder.*

**Problem 5 [4 points]**

For any  $\alpha \in \mathbb{R}$  one can define the following generalization of the binomial coefficient:

$$\binom{\alpha}{k} := \frac{\alpha(\alpha - 1)(\alpha - 2) \cdots (\alpha - k + 1)}{k!}$$

for  $k \geq 1$ , and  $\binom{\alpha}{0} := 1$ . With that definition, let us define the function

$$f(x) = \sum_{k=0}^{\infty} \binom{\alpha}{k} x^k$$

on the interval  $(-1, 1)$ .

(a) Show that

$$f'(x) = \frac{\alpha}{1+x} f(x).$$

Justify the steps of your argument.

(b) Show that

$$g(x) = \frac{f(x)}{(1+x)^\alpha}$$

equals the constant 1.

(c) Conclude that  $f(x) = (1+x)^\alpha$  on  $(-1, 1)$ .