Constructor University — March 7, 2023
Spring 2023

Advanced Calculus and Methods of Mathematical Physics

Homework 5

Due on March 14, 2023, before the tutorial.

Problem 1 [5 points]

(a) Let $U \subset \mathbb{R}^n$ and $V \subset \mathbb{R}^m$ be open and let $f : U \to V$ be differentiable at $p \in U$ and $g : V \to \mathbb{R}^j$ be differentiable at $f(p)$. Prove that then the composition $F := g \circ f : U \to \mathbb{R}^j$ is differentiable at p with derivative $DF|_p = Dg|_{f(p)}\, Df|_p$.

(b) Now check the chain rule for a specific example. Let $g(x, y) := e^{-x^2-y^2}$ and $f(r, \varphi) := (r\cos\varphi, r\sin\varphi)$ and compute first $DF|_p$ directly, and then $Dg|_{f(p)}\, Df|_p$, for any $p = (r, \varphi)$.

Problem 2 [3 points]
Let $h : [0, \infty) \times [0, 2\pi) \times [0, \pi] \to \mathbb{R}^3$ be defined as

$$h(r, \varphi, \theta) := (r\cos\varphi\sin\theta, r\sin\varphi\sin\theta, r\cos\theta).$$

This is the change from spherical to Cartesian coordinates. Compute the Jacobian matrix of h and its determinant.

Problem 3 [3 points]
Show that the function

$$u : \mathbb{R}^2 \setminus \{(0, 0)\} \to \mathbb{R},\ x \mapsto \ln \|x\|$$

solves the Laplace equation

$$\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right) u(x, y) = 0.$$

(Here, $\|x\| = \sqrt{x^2 + y^2}$ is the usual 2-norm.)

Problem 4 [5 points]
Consider the function $f : \mathbb{R}^2 \to \mathbb{R}$ given by $f(x, y) = xy(x^2 - y^2)/(x^2 + y^2)$ for $(x, y) \neq (0, 0)$.

(a) Is f twice partially differentiable on $\mathbb{R}^2 \setminus (0, 0)$, and are the second derivatives continuous?

(b) Show that with $f(0, 0) = 0$ the function f is twice partially differentiable at $(0, 0)$.

(c) Compute $\dfrac{\partial^2 f}{\partial x\,\partial y}$ and $\dfrac{\partial^2 f}{\partial y\,\partial x}$ at $(x, y) = (0, 0)$. Should that be surprising?

Problem 5 [4 points]

Consider the function $f\colon \mathbb{R}^2 \to \mathbb{R}$ given by $f(x, y) = e^{-y^2} - x^2(y + 1)$.

(a) Prove that $f \in C^2$ and write down the Taylor expansion to second order.

(b) Does f have local extrema? Are these also global extrema?

Bonus Problem [4 points]: Banach fixed-point theorem

In any metric space (X, d), a map $f : X \to X$ is called a *contraction* if there is an $0 \leq r < 1$ such that $d(f(x), f(y)) \leq rd(x, y)$ for all $x, y \in X$. Prove the following theorem called *Banach fixed-point theorem* or *contraction mapping principle*: If (X, d) is a complete metric space, then any contraction $f : X \to X$ has a unique fixed point (i.e., a unique $x^* \in X$ with $f(x^*) = x^*$).

Hint: Uniqueness is easy. The fixed point can be constructed by defining a sequence $x_{n+1} = f(x_n)$. *Is this a Cauchy sequence?*