Constructor University | April 11, 2023
Spring 2023

Advanced Calculus and Methods of Mathematical Physics

Homework 9

Due on April 18, 2023, before the tutorial.

Problem 1 [3 points]
Let $R > 0$.

(a) Let $F \in C^1([0, R^2])$ and set $f = F'$. Show that

$$\int_{B(0,R)} f(x^2 + y^2)\ \mathrm{d}S = \pi\left(F(R^2) - F(0)\right).$$

(b) Calculate $\displaystyle\int_{B(0,R)} \cos(x^2 + y^2)\ \mathrm{d}S$.

(c) Calculate $\displaystyle\int_{B(0,R)} \exp(-x^2 - y^2)\ \mathrm{d}S$.

Problem 2 [3 points]
Let

$$D = \{(x, y) \in \mathbb{R}^2 \colon 1 \leq x^2 - y^2 \leq 9, 0 \leq x \leq 4, y \geq 0\}.$$

Compute

$$\int_D x\, y\, e^{x^2 - y^2}\ \mathrm{d}S.$$

Problem 3 [4 points]

(a) Use the spherical coordinates from Problem 2 Homework 5 to compute the volume of the unit sphere.

(b) Let $0 < a < b$ and

$$D = \{x \in \mathbb{R}^3 \colon x_i \geq 0, a \leq \|x\| \leq b\}.$$

For $p, q \in \mathbb{R}$ with $q \geq 0$, compute

$$A_i = \int_D x_i^q\, \|x\|^p\ \mathrm{d}x.$$

Problem 4 [3 points]
Calculate the volume of the domain in $\mathbb{R}^3$ bounded by the surfaces $z = x^2 + y^2$, $y = x^2$, $y = 1$, and $z = 0$.

Problem 5 [3 points]
Recall the definition for a line integral of a vector field $F \in C(D, \mathbb{R}^n)$ on a domain $D \subset \mathbb{R}^n$ along a smooth curve $\gamma \in C^1([a, b], D)$,

$$\int_\gamma F \cdot \mathrm{d}x = \int_a^b F(\gamma(t)) \cdot \gamma'(t)\ \mathrm{d}t\,.$$

Prove, by explicit calculation, that this definition is independent of the choice of parametrization of the curve.

Problem 6 [4 points]
(Kantorovitz, Exercises 4.3.15, Problem 1.) Let γ be the helix parameterized by

$$\gamma(t) = (a\cos t, a\sin t, bt)$$

with $a, b > 0$.

(a) Find the arc length $s(t)$ of the arc $\{\gamma(\tau)\colon 0 \le \tau \le t\}$.

(b) Find the length of one turn of the helix.

(c) Let $F = (-y, x, z)$ be a vector field in $\mathbb{R}^3$. Calculate the line integral

$$\int_\gamma F \cdot \mathrm{d}x$$

over the turn of the helix $t \in [0, 2\pi]$.

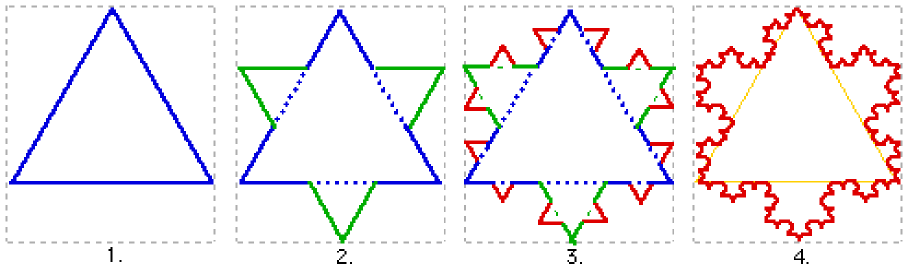
(d) Calculate the line integral

$$\int_\gamma \frac{1}{\|x\|}\ \mathrm{d}s$$

over the same turn of the helix.

Bonus Problem [4 points]
Let us consider the von-Koch curve ("snowflake"). It is constructed iteratively in the following way:

1. Start with an equilateral triangle with sides of unit length.
2. On the middle third of each of the three sides, build an equilateral triangle with sides of length 1/3. Erase the base of each of the three new triangles.
3. On the middle third of each of the twelve sides, build an equilateral triangle with sides of length 1/9. Erase the base of each of the twelve new triangles.



The boundary after the n-th step is a piecewise linear curve γ_n; parametrize it as a map $\gamma_n \colon [0,1] \to \mathbb{R}^2$ so that $\|\gamma_n'\|$ is constant.

(a) How many pieces does each γ_n have and what is their total length?

(b) Show that the sequence of maps γ_n converges uniformly to a continuous map $\gamma \colon [0,1] \to \mathbb{R}^2$.

(c) Show that γ is not rectifiable.