Constructor University
Spring 2023

April 18, 2023

Advanced Calculus and Methods of Mathematical Physics

Homework 10

Due on April 25, 2023, before the tutorial.

Problem 1 [8 points]

(a) Compute the volume of the ice cream cone defined by $0 \le \varphi \le 2\pi$, $0 \le \theta \le \frac{\pi}{4}$, $0 \le r \le \sqrt{2}$ (in the notation of Problem 2 from Homework 5).

(b) Compute $\int_B e^{(x^2+y^2+z^2)^{3/2}} dS$, where B is the unit ball.

(c) The function

$$g : [0, \infty) \times [0, 2\pi) \times \mathbb{R} \to \mathbb{R}^3, (r, \phi, z) \mapsto (r \cos\varphi, r \sin\varphi, z)$$

describes the change from cylindrical coordinates in $\mathbb{R}^3$ to Cartesian coordinates. Let $f : [a, b] \to \mathbb{R}, y = f(z)$ be continuous and positive except possibly at the endpoints. Denote by $\overline{D} \subset \mathbb{R}^3$ the closed domain obtained from rotating the graph of f about the z-axis. Use cylindrical coordinates to compute the volume of $\overline{D}$.

Problem 2 [5 points]

(Kantorovitz, Exercises 4.3.15, Problem 3.) The curve $\gamma \subset \mathbb{R}^2$ is given in polar coordinates by the C^1 function

$$r = g(\theta)\,, \quad \theta \in [a, b]\,.$$

(a) Show that the arc length function on γ is given by

$$s(\theta) = \int_a^\theta \sqrt{g'(\tau)^2 + g(\tau)^2}\,\mathrm{d}\tau\,.$$

(b) For $g(\theta) = 1 - \cos\theta$, $[a, b] = [0, 2\pi]$, find the length of the curve γ.

Problem 3 [5 points]

(Kantorovitz, Exercises 4.3.15, Problem 4.) Let γ be the circle centered at the origin with radius r. Calculate

$$\int_\gamma F \cdot \mathrm{d}x$$

for

(a) $F(x) = (x_1 - x_2, x_1 + x_2)$,

(b) $F(x) = \nabla\phi$ with $\phi(x) = \arctan(x_2/x_1)$.

Explain the difference between case (a) and case (b) in light of the theory discussed in class.

Problem 4 [2 points]

Let γ be the ellipse with equation

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1\,,$$

with counter-clockwise orientation. Let

$$F(x, y) = e^x(\sin y, \cos y)\,.$$

Calculate

$$\int_\gamma F \cdot \mathrm{d}x\,.$$