

Advanced Calculus and Methods of Mathematical Physics

Homework 11

Due on May 2, 2023, before the tutorial.

Problem 1 [6 points]

(Kantorovitz, Exercises 4.4.5, Problems 1, 2, 3.) Find the area $S(D)$ of the domain $D \subset \mathbb{R}^2$ with given boundary $\partial D = \gamma$ for the cases where:

(a) γ is the *cardioid* parameterized by

$$\gamma(t) = (1 - \cos t)(\cos t, \sin t), \quad t \in [0, 2\pi].$$

(b) γ is the *hypocycloid* satisfying the equation

$$\left(\frac{x}{a}\right)^{\frac{2}{3}} + \left(\frac{y}{b}\right)^{\frac{2}{3}} = 1$$

with $a, b > 0$. (*Hint*: Find a trigonometric parametrization.)

(c) γ is the curve with parametrization

$$\gamma(t) = (t^2 - 1)(1, t), \quad t \in [-1, 1].$$

Problem 2 [7 points]

(From Moskowitz/Paliogiannis, p. 460.) Evaluate the following line integrals along a curve γ , both directly and by using Green's theorem:

(a) $\int_{\gamma} [(1 - x^2)y \, dx + (1 + y^2)x \, dy]$, where γ is the unit circle in anti-clockwise orientation.

(b) $\int_{\gamma} [x y^2 \, dy - x^2 y \, dx]$, where γ is the boundary of the annulus $1 \leq x^2 + y^2 \leq 4$, in standard orientation (i.e., counter-clockwise for the outer circle and clockwise for the inner circle).

(c) $\int_{\gamma} [(y - \sin x) \, dx + \cos x \, dy]$, where γ is the perimeter of the triangle with vertices $(0, 0)$, $(\frac{\pi}{2}, 0)$, and $(\frac{\pi}{2}, 1)$ in counter-clockwise orientation.

Problem 3 [7 points]

Prove the following identities for vectors $\mathbf{a}, \mathbf{b}, \mathbf{c}, \mathbf{d} \in \mathbb{R}^3$.

(a) The “*BAC–CAB-identity*”

$$\mathbf{a} \times (\mathbf{b} \times \mathbf{c}) = \mathbf{b}(\mathbf{a} \cdot \mathbf{c}) - \mathbf{c}(\mathbf{a} \cdot \mathbf{b}).$$

(b) The *Jacobi identity* in three dimensions

$$\mathbf{a} \times (\mathbf{b} \times \mathbf{c}) + \mathbf{b} \times (\mathbf{c} \times \mathbf{a}) + \mathbf{c} \times (\mathbf{a} \times \mathbf{b}) = \mathbf{0}.$$

(c) The *Cauchy–Binet formula* in three dimensions

$$(\mathbf{a} \times \mathbf{b}) \cdot (\mathbf{c} \times \mathbf{d}) = (\mathbf{a} \cdot \mathbf{c})(\mathbf{b} \cdot \mathbf{d}) - (\mathbf{a} \cdot \mathbf{d})(\mathbf{b} \cdot \mathbf{c}).$$

Then, show that $\|\mathbf{a} \times \mathbf{b}\| = \|\mathbf{a}\|\|\mathbf{b}\|\sin\theta$ (which equals the area of a parallelogram spanned by \mathbf{a} and \mathbf{b}), where θ is the angle between \mathbf{a} and \mathbf{b} .