Constructor University — May 9, 2023
Spring 2023

Advanced Calculus and Methods of Mathematical Physics

Homework 13

This homework is not for credit and cannot be handed in for grading. Solution sketches can be found on the website. The exercises on this homework sheet are not relevant for the regular exam questions, but might be relevant for a bonus question.

Problem 1

Suppose $f, g\colon [0, 2\pi] \to \mathbb{C}$ are Riemann-integrable. Prove the following properties of the Fourier transform.

(a) If f is real-valued, then $f_{-k} = \overline{f_k}$.

(b) If f is continuously differentiable, then $(f')_k = ik\, f_k$.

(c) Let $T_a f$ denote the right-translation by $a \in \mathbb{R}$ of f, i.e., $(T_a f)(x) = f(x - a)$, with the understanding that f is periodically extended outside of its fundamental domain $[0, 2\pi]$. Show that

$$(T_a f)_k = e^{-ika}\, f_k\,.$$

(d) Let

$$(f * g)(x) = \int_0^{2\pi} f(y)\, g(x - y)\, \mathrm{d}y$$

denote the *convolution* of the functions f and g, again with the understanding that the functions are periodically extended outside of their fundamental domain. Show that

$$(f * g)_k = 2\pi\, f_k\, g_k\,.$$

Problem 2

Compute the Fourier transform of the “saw-tooth function” $f(x) = x$ on $[-\pi, \pi)$, periodically extended outside its fundamental domain.

Problem 3

(a) Compute the Fourier series of the function $f(x) = (x - \pi)^2$ for $x \in [0, 2\pi]$ (Example B from Class Session 25).

(b) Finish the proof that we started in class of the fact that the bump function from Example A (Session 25) is mean-square convergent by showing that

$$\sum_{k=-\infty}^{\infty} |\hat{f}_k|^2 = \frac{a}{2\pi}.$$

Problem 4

Which of the following functions are complex-differentiable?

(a) $f(z) = z^2$,

(b) $f(z) = |z|^2$,

(c) $f(z) = \cos(z)$.

Problem 5

Show that if $f(z) = u(x,y) + i\,v(x,y)$ with $z = x + iy$ is complex-differentiable (holomorphic) on some domain $D \subset \mathbb{C}$, then u and v are harmonic functions on $\mathbb{R}^2$, i.e., $\Delta u = \Delta v = 0$, where $\Delta = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}$ is the Laplace operator on the corresponding domain of $\mathbb{R}^2$.

Remark: That adds another item to the list of fundamental equivalences in complex variable calculus: "harmonic" $\iff$ "complex-differentiable (holomorphic)" $\iff$ "convergent Taylor series (analytic)" $\iff$ "path independence of complex line integral (identification with conservative vector field)".

Problem 6

Use the residue theorem to compute

$$\int_{|z|=1} z^2 \sin\frac{1}{z}\,\mathrm{d}z\,.$$