Advanced Calculus and Methods of Mathematical Physics — Session 1

Prof. Sören Petrat, Constructor University, Spring 2023 — Feb. 3, 2023

Organization:

- Prof. Sören Petrat (Mathematics)
  Office: room 112 in Research I
- class: Tue 11:15 – 12:30, Fri 8:15 - 9:30, in-person, Res. I lecture hall
  (note: software lab Fri 9:45 - 11:00 taught by Ulrich Kleinekathöfer)
- class website: news, syllabus, lecture notes, references, homework sheets
- grade:
  - 100 % final exam
  - bonus: up to 10% from homework sheets (see website for details)
    (Note: bonus cannot change "fail" to "pass" grade.)
  - note: the total module grade is $\frac{2}{3}$ this class and $\frac{1}{3}$ the lab
- TA: Abdullah Irfan Basheer
- homework sheets:
  - weekly, starting next week
  - hand in before class
  - solutions discussed in weekly tutorial ← not mandatory, but highly recommended
- books: mostly Kantorovitz – Several Real Variables
  (see class schedule on website for more references)
- style of this class:
  - in between "Calculus and Linear Algebra I" and "Analysis I"
  - includes proofs, but avoids too much abstraction

Topics:

- Sequences and series of functions
- Differentiation in many variables → ~ $\frac{1}{3}$ of class
- Integration in many variables → ~ $\frac{1}{3}$ of class
- Fourier series/transform
- Complex analysis

1. Sequences and Series of Functions

1.1 Review of differentiation, integration, and Taylor's theorem

In this chapter, we consider functions $f: D \to \mathbb{R}$, with $D \subset \mathbb{R}$ (usually D is an interval or $D = \mathbb{R}$).

Let us recall a few important properties ($D \subset \mathbb{R}$ open):

- f is **continuous** at $\tilde{x} \in D$

  $:\Longleftrightarrow$ $\underbrace{\forall}_{\text{"for all"}}$ sequences $(x_n)_{n\in\mathbb{N}}$ in D with $x_n \xrightarrow{n\to\infty} \tilde{x}$, we have $f(x_n) \xrightarrow{n\to\infty} f(\tilde{x})$ $\left(\lim_{n\to\infty} f(x_n) = f(\tilde{x})\right)$.

  We write this as $\lim_{x\to\tilde{x}} f(x) = f(\tilde{x})$.

  $\Longleftrightarrow f(\tilde{x}+h) = f(\tilde{x}) + R_{\tilde{x}}(h)$ with $\lim_{h\to 0} R_{\tilde{x}}(h) = 0$

  $\Longleftrightarrow \forall \varepsilon > 0 \ \underbrace{\exists}_{\text{"there exists"}} \delta > 0$ s.t. $\forall x \in D: |x-\tilde{x}| < \delta$ implies $|f(x) - f(\tilde{x})| < \varepsilon$

- f is **differentiable** at $\tilde{x} \in D$

  $\Longleftrightarrow f'(\tilde{x}) := \lim_{h\to 0} \dfrac{f(\tilde{x}+h) - f(\tilde{x})}{h}$ exists

  note: from this def. we see immediately that differentiability implies continuity

  $\Longleftrightarrow \exists\, m \in \mathbb{R}$ s.t. $f(\tilde{x}+h) = f(\tilde{x}) + mh + R(h)$ with $\lim_{h\to 0} \dfrac{R(h)}{h} = 0$

  (then $m = f'(\tilde{x})$ is the derivative of f at $\tilde{x} \in D$)

(Note: $f: D \to \mathbb{R}$ cont./diff.able $:\Longleftrightarrow f: D \to \mathbb{R}$ cont./diff.able $\forall \tilde{x} \in D$.)

Recall a few standard results:

- f differentiable at $\tilde{x} \Rightarrow f$ continuous at $\tilde{x}$

  (but converse does not hold; there are even everywhere continuous and nowhere differentiable fct.s)

- product (or Leibniz) rule and chain rule

- mean-value thm.: let $f:[a,b]\to\mathbb{R}$ be cont. and differentiable on (a,b). Then there is a $z\in(a,b)$ with

$$f'(z)=\frac{f(b)-f(a)}{b-a}$$

[draw a picture to visualize this]