Advanced Calculus and Methods of Mathematical Physics   Session 2

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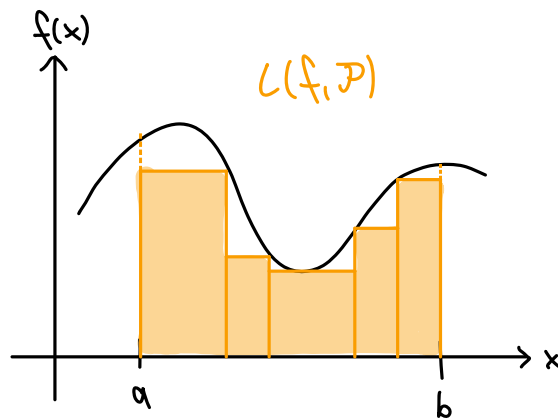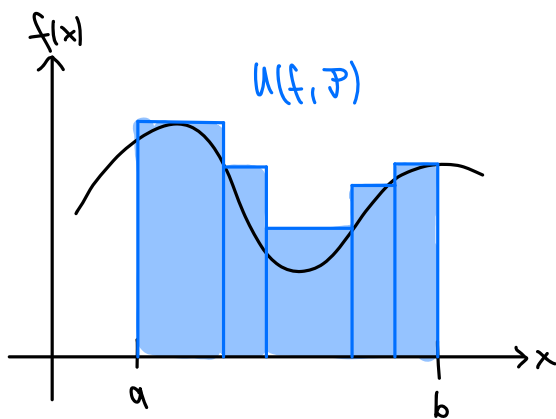
For the Riemann integral, let us consider $f: [a,b] \to \mathbb{R}$ (bounded). We define:

- a partition $\mathcal{P}$ of $[a,b]$ is a finite set of points $x_1, \ldots, x_n$ s.t. $a = x_0 \leq x_1 \leq \ldots \leq x_{n-1} \leq x_n = b$;

- the upper Riemann sum is $U(f, \mathcal{P}) := \sum_{i=1}^{n} \left[\sup_{x \in [x_{i-1}, x_i]} f(x) \right] (x_i - x_{i-1})$;

- the lower Riemann sum is $L(f, \mathcal{P}) := \sum_{i=1}^{n} \left[\inf_{x \in [x_{i-1}, x_i]} f(x) \right] (x_i - x_{i-1})$;

- the upper Riemann integral $\overline{\int_a^b} f(x) dx := \inf_{\mathcal{P}} U(f, \mathcal{P})$ ← always defined

- the lower Riemann integral $\underline{\int_a^b} f(x) dx := \sup_{\mathcal{P}} L(f, \mathcal{P})$ ← always defined

[do you recall a counter example?
i.e., a function that is not Riemann integrable]

Then: $f: [a,b] \to \mathbb{R}$ is **Riemann integrable** if upper and lower Riemann integrals coincide.

Then $\int_a^b f \equiv \int_a^b f(x) dx := \underline{\int_a^b} f(x) dx = \overline{\int_a^b} f(x) dx.$

$\equiv \int_a^b dx\, f(x)$

Recall:

- integration by parts, integration by substitution
- $f: [a,b] \to \mathbb{R}$ continuous $\Rightarrow f$ integrable

- **Fundamental Theorem of Calculus:**

  – version 1: Let f be integrable on $[a,b]$, F cont. on $[a,b]$ and differentiable on (a,b) with
  $$F'(x) = f(x) \quad \forall x \in (a,b) \text{ (such } F \text{ are called anti-derivative of } f). \text{ Then}$$
  $$\int_a^b f(x)\,dx = F(b) - F(a).$$

  – version 2: Let f be integrable on $[a,b]$ and continuous at $\tilde{x} \in (a,b)$. Define $F: [a,b] \to \mathbb{R}$ by
  $$F(x) - F(a) := \int_a^x f(t)\,dt. \text{ Then } F \text{ is continuous on } [a,b] \text{ and differentiable}$$
  at $\tilde{x}$ with $F'(\tilde{x}) = f(\tilde{x})$. [can you prove this?]

Let us recall the following notation. Let $f: I \to \mathbb{R}$, I an open interval. Then:

f is **of class $C^n(I)$** if f is n times differentiable and all derivatives $f', f'', \ldots, f^{(n)}$ are cont.

Next: Taylor series. Let $f: I \to \mathbb{R}$, I an open interval.

We try to approximate $f(x)$ near $c \in I$ by polynomials s.t. the derivatives up to some order agree in c.

For f n times differentiable, we write $f(x) = P_{n,c}(x) + R_{n,c}(x)$ with
$$P_{n,c}(x) := \sum_{k=0}^n \frac{f^{(k)}(c)}{k!}(x-c)^k \quad \text{the } n\text{-th order Taylor polynomial about } c, \text{ and}$$

$R_{n,c}(x)$ the n-th order Taylor remainder.

[check that indeed $P_{n,c}^{(\ell)}(c) = f^{(\ell)}(c) \quad \forall \ell = 0, \ldots, n$]

Ex.: For $f(x)=e^x$, we have $P_{n,0}(x) = \sum_{k=0}^{n} \frac{x^k}{k!}$.

Note: We know that $\lim\limits_{x \to c} \frac{R_{n,c}(x)}{(x-c)^n} = 0$ (e.g., by using L'Hôpital n times).

[check this]

Notation: $R_{n,c}(x) = o((x-c)^n)$ as $x \to c$. If $\boxed{h(x) = o(g(x))}$ as $x \to c$ ($c = \pm \infty$ possibly), we say

$h(x)$ is "of lower order than $g(x)$" as $x \to c$ (or "little oh of g of x") if

$\frac{h(x)}{g(x)} \to 0$ as $x \to c$.

Note: There is also the "big oh" notation: $\boxed{h(x) = O(g(x))}$ if $\left| \frac{h(x)}{g(x)} \right| \le$ constant as $x \to c$.

↳ "h is of the order of g"