Advanced Calculus and Methods of Mathematical Physics   Session 3

Prof. Sören Petrat, Constructor University, Spring 2023   Feb. 10, 2023

last time: For $f \in C^{(n)}$, we write $f(x) = \sum_{k=0}^{n} \frac{f^{(k)}(c)}{k!}(x-c)^k + R_{n,c}(x)$.

$$\underbrace{\phantom{\sum_{k=0}^{n} \frac{f^{(k)}(c)}{k!}(x-c)^k}}_{=: P_{n,c}(x)} \quad \underbrace{\phantom{R_{n,c}}}_{o((x-c)^k)}$$

Note: Taylor polynomials are unique: Let f be n times differentiable, and $c \in I$. If Q is a polynomial

of degree $\leq n$ with $f(x) - Q(x) = o((x-c)^n)$ as $x \to c$, then $P_{n,c}(x) = Q(x)$.

$\implies$ We can use any method to find a polynomial Q of degree $\leq n$ with

$f(x) = Q(x) + o((x-c)^n)$; then $Q(x)$ is always the Taylor polynomial.

(Useful, e.g., for products or compositions of fct.s)   ($\rightarrow$ see HW 1, Problem 3)

So far we have not made the remainder $R_{n,0}(x)$ very explicit. But using the fundamental

thm. of calculus we can write:

$$f(x) - f(c) = \underbrace{\int_c^x f'(t)\,dt}$$

int. by parts with $(t-x)$ as antiderivative of 1 (w.r.t. t)

$$= \int_c^x 1 \cdot f'(t)\,dt \;\downarrow\; = \underbrace{(t-x)f'(t)\Big|_c^x}_{= (x-c)f'(c)} - \int_c^x (t-x)f''(t)\,dt$$

$$\implies f(x) = f(c) + f'(c)(x-c) + \int_c^x (x-t)f''(t)\,dt$$

[check the next order or do a proof by induction]

Repeating this yields Taylor's thm.:

Let $f \in C^{(n+1)}(I)$, $I = (a,b)$, $c \in I$. If $x \in I$, then

$$R_{n,c}(x) = \int_c^x \frac{(x-t)^n}{n!} f^{(n+1)}(t)\,dt \qquad \text{(integral form of remainder)}.$$

This can be used for rigorous remainder estimates, e.g. :

- If additionally there is a $C > 0$ s.t. $\left|f^{(n+1)}(x)\right| \le C \; \forall x \in I$, then

$$|R_{n,c}(x)| \le C \, \frac{|x-c|^{n+1}}{(n+1)!} = C \, \frac{|x-c|}{n+1} \cdot \frac{|x-c|}{n} \cdots \frac{|x-c|}{1} \xrightarrow{n \to \infty} 0 \quad \text{for any given } x, c.$$

So in this case $f(x) = \lim\limits_{n \to \infty} \sum\limits_{k=0}^{n} \frac{f^{(k)}(c)}{k!} (x-c)^k \quad$ for all $x \in I$

Note: There are also the Cauchy and Lagrange forms of the remainder.
(see HW 1 Problem 2 for the Lagrange form)
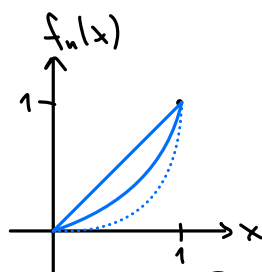
1.2 Sequences of Functions

Let $\left(f_n(x)\right)_{n \in \mathbb{N}}$ be a sequence of functions $f_n : D \to \mathbb{R}$, $D \subset \mathbb{R}$.

Things we want to know (here $f(x) := \lim\limits_{n \to \infty} f_n(x)$ for all $x \in D$):

- If all f_n are continuous, is f also continuous?
- Is $\lim\limits_{n \to \infty} \int f_n(x) dx = \int \lim\limits_{n \to \infty} f_n(x) \, dx =: \int f(x) dx$?
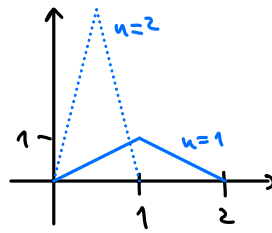- Is $\lim\limits_{n \to \infty} f_n'(x) = f'(x)$?

Examples:

- $f_n(x) = x^n$, $x \in [0,1]$

Each f_n is continuous, but $\lim\limits_{n \to \infty} f_n(x) = \begin{cases} 0 & \text{for } x \in [0,1) \\ 1 & \text{for } x = 1 \end{cases}$ is not continuous.

- $f_n(x) = \begin{cases} n^2 x, & 0 \le x \le \frac{1}{n} \\ 2n - n^2 x, & \frac{1}{n} \le x \le \frac{2}{n} \\ 0, & \frac{2}{n} \le x \le 1 \end{cases}$

Here, $\int_0^2 f_n(x)\,dx = \ldots = 1$ for all $n \in \mathbb{N}$, but $\lim\limits_{n \to \infty} f_n(x) = 0$ for all $x \in [0,2]$.

So $\lim\limits_{n \to \infty} \int_0^2 f_n(x)\,dx \ne \int_0^2 f(x)\,dx$.

- $f_n(x) = \frac{1}{n} \sin(nx)$.

Here, $f_n(x) \xrightarrow{n \to \infty} 0 = f(x)$, but $f_n'(x) = \cos(nx)$ does not have a limit $\forall x$ as $n \to \infty$.

Key concept to achieve nicer convergence results: uniform convergence

Above, we only used the notion of pointwise convergence:

$(f_n)_n$ converges to $f: D \to \mathbb{R}$ pointwise $:\Longleftrightarrow$ $f_n(x) \xrightarrow{n \to \infty} f(x)$ for all $x \in D$.

In other words:

For all $x \in D$ we have: $\forall \varepsilon > 0 \; \exists N \in \mathbb{N}$ s.t. $\forall n \ge N: |f_n(x) - f(x)| < \varepsilon$

Here, x is choosen first, and ε and N may depend on the choice of x.

Next we require ε and N that work for all x at the same time:

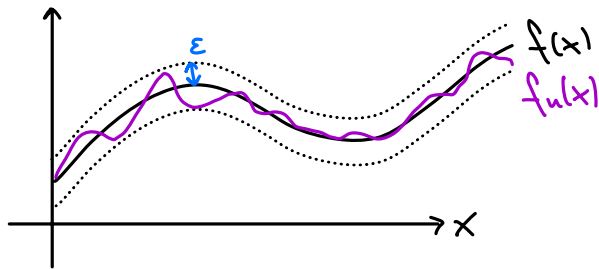# Definition:

$(f_n)_n$ with $f_n: D \to \mathbb{R}$, $D \subset \mathbb{R}$ [converges uniformly] to $f: D \to \mathbb{R}$ if

$\forall \varepsilon > 0 \ \exists N \in \mathbb{N}$ s.t. $\forall n \geq N$ and $\forall x \in D$: $|f_n(x) - f(x)| < \varepsilon$.

here, the N large enough is independent of x

Graphically:

$f(x)$
$f_n(x)$

"all f_n's are in any ε-tube around f (for n large enough)"

For the examples $f_n(x) = x^n$, $x \in [0,1]$, and $f_n(x) = \begin{cases} n^2 x & 0 \leq x \leq \frac{1}{n} \\ 2n - n^2 x & \frac{1}{n} \leq x \leq \frac{2}{n} \\ 0 & \frac{2}{n} \leq x \leq 1 \end{cases}$

we do not have uniform convergence!