Advanced Calculus and Methods of Mathematical Physics   Session 4

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Recall: Let $f_n : D \overset{\subset \mathbb{R}}{\to} \mathbb{R}$, $f : D \to \mathbb{R}$.

- f_n converges to f pointwise means:

  Fix $x \in D$ first. Then: $\forall \varepsilon > 0 \; \exists N \in \mathbb{N}$ s.t. $|f_n(x) - f(x)| < \varepsilon \; \forall n \geq N$.

  Here: $|f_n(x) - f(x)| \overset{n \to \infty}{\longrightarrow} 0$ for each fixed x.

  But $\sup_{x \in D} |f_n(x) - f(x)|$ might not go to 0.

- $\boxed{f_n \text{ converges to } f \text{ uniformly}}$ means:

  $\forall \varepsilon > 0 \; \exists N \in \mathbb{N}$ s.t. $\forall x \in D : |f_n(x) - f(x)| < \varepsilon \; \forall n \geq N$.

  <u>these ε and N have to work for all x at the same time</u>

  Here, $\sup_{x \in D} |f_n(x) - f(x)| \overset{n \to \infty}{\longrightarrow} 0$.

Then indeed we have the following results:

Theorem:

Let $(f_n)_{n \in \mathbb{N}}$ be a sequence of functions with $f_n : D \to \mathbb{R}$, $D \subset \mathbb{R}$, and $f : D \to \mathbb{R}$. Then:

a) If $f_n \to f$ uniformly and all f_n are continuous, then f is continuous.

b) If $f_n \to f$ uniformly, $D = [a,b]$ for some $a < b$ and all f_n are Riemann integrable, then f is Riemann integrable and $\lim_{n \to \infty} \int_a^b f_n = \int_a^b f$.

c) If $D = [a,b]$ for some $a < b$ and all f_n are $C^{(1)}$ (continuously differentiable), $g : D \to \mathbb{R}$, and $f_n \to f$ pointwise, $f_n' \to g$ uniformly, then g is continuous and f is $C^{(1)}$ with $f' = g$.

<u>Proof of b):</u> If $f_n \to f$ uniformly, then $\varepsilon_n := \sup\limits_{x \in [a,b]} |f_n(x) - f(x)| \xrightarrow{n \to \infty} 0$.

So for any $n \in \mathbb{N}$: $\quad f_n - \varepsilon_n \leq f \leq f_n + \varepsilon_n \quad$ (these inequalities are meant pointwise)

$$\Rightarrow \quad \int_a^b (f_n - \varepsilon_n) \leq \underline{\int_a^b} f \leq \overline{\int_a^b} f \leq \int_a^b (f_n + \varepsilon_n), \quad (*)$$

so $\quad 0 \leq \overline{\int_a^b} f - \underline{\int_a^b} f \leq 2(b-a)\varepsilon_n \xrightarrow{n \to \infty} 0 \quad$, so f Riemann integrable.

$$\Rightarrow \overline{\int_a^b} f = \underline{\int_a^b} f = \int_a^b f \quad \text{and} \quad \left| \int_a^b f - \int_a^b f_n \right| \leq (b-a)\varepsilon_n \xrightarrow{n \to \infty} 0 \quad, \text{so} \quad \int_a^b f_n \to \int_a^b f. \quad \square$$

in $(*)$

Note:

- The theorem in particular applies to $S_n := \sum\limits_{k=0}^{n} f_k$, e.g., if $\sum\limits_{k=0}^{n} f_k \xrightarrow{n \to \infty} S \equiv \sum\limits_{k=0}^{\infty} f_k$

  called "partial sums"

  uniformly and all f_k Riemann integrable, then $\sum\limits_{k=0}^{\infty} \int_a^b f_k = \int_a^b \sum\limits_{k=0}^{\infty} f_k$.

- One can show: If $|f_k(x)| \leq M_k \ \forall k \ \forall x$ with $\sum\limits_{k=0}^{\infty} M_k$ convergent, then $\sum\limits_{k=0}^{n} f_k$ converges uniformly to f, def. pointwise by $f(x) = \sum\limits_{k=0}^{\infty} f_k(x)$. (Weierstrass M-test)

1.3 Power Series

First, a quick review on convergence of series.

Let $(a_k)_n$ be a sequence in $\mathbb{R}$ and $S_n := \sum_{k=0}^{n} a_k$ the partial sums. Then:

- $\sum_{k=0}^{\infty} a_k$ exists $:\Longleftrightarrow \lim_{n \to \infty} \sum_{k=0}^{n} a_k$ exists
- $\sum_{k=0}^{\infty} a_k$ converges absolutely $:\Longleftrightarrow \sum_{k=0}^{\infty} |a_k|$ converges
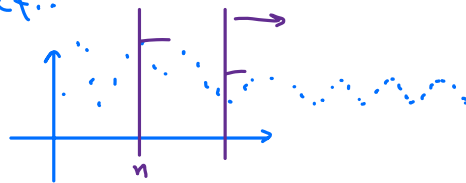
The most common convergence tests are:

- **Comparison test:** Fix some $N \in \mathbb{N}$. Then:
  - If $|a_k| \leq b_k \ \forall k \geq N$ and Σb_k conv., then Σa_k conv.
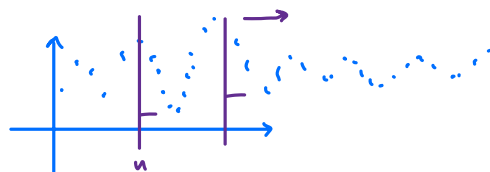 - If $a_k \geq c_k \geq 0 \ \forall k \geq N$ and Σc_k diverges, then Σa_k diverges.

Note: For a sequence (a_k) in $\mathbb{R}$ we def.:

- $\limsup_{k \to \infty} a_k := \inf_{n \in \mathbb{N}} \left(\sup_{k \geq n} a_k \right)$

- $\liminf_{k \to \infty} a_k := \sup_{n \in \mathbb{N}} \left(\inf_{k \geq n} a_k \right)$

E.g., $\lim_{k \to \infty} (-1)^k$ does not exist, but $\limsup_{k \to \infty} (-1)^k = 1$ and $\liminf_{k \to \infty} (-1)^k = -1$.

If $\liminf_{k \to \infty} a_k = \limsup_{k \to \infty} a_k$, then $\lim_{k \to \infty} a_k$ exists.

- **Root test**: Define $L := \limsup\limits_{k \to \infty} \sqrt[k]{|a_k|}$. Then:

  - If $L < 1$: $\sum a_k$ conv.,
  - If $L > 1$: $\sum a_k$ diverges,
  - If $L = 1$ the test is inconclusive.

  Can be applied to any sequence! (no specific conditions like in the other tests)

- **Ratio test**: If $\limsup\limits_{k \to \infty} \left| \frac{a_{k+1}}{a_k} \right| < 1$, then $\sum a_k$ conv.

- **Integral test**: Let $f: [0, \infty) \to [0, \infty]$ be Riemann integrable on $[0, b]$ $\forall b > 0$ and monotonically decreasing. Then $\sum\limits_{k=0}^{\infty} f(k)$ exists if and only if $\int\limits_0^{\infty} f(x) dx$ exists.

- **Leibniz criterion** ("alternating series test"): Let $a_k \geq 0$ and (a_k) monotonically decreasing with $a_k \xrightarrow{k \to \infty} 0$. Then $\sum (-1)^k a_k$ converges.

Examples: see homework