Advanced Calculus and Methods of Mathematical Physics — Session 6

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1.4 Metric and Normed Spaces

Let us take an abstract point of view for a moment.

Given some set X, there are different possibilities to define "closeness" for elements of X:

- The most general/abstract type of space is a $\boxed{\text{topological space}}$ (X,τ), where τ is a collection of subsets of X satisfying the axioms:
  - $\emptyset \in \tau$, $X \in \tau$,
  - arbitrary unions of elements of τ belong to τ,
  - finite intersections of elements of τ belong to τ.

Elements of τ are called $\boxed{\text{"open sets"}}$. Complements of open sets are called "closed sets".

For example, in $\mathbb{R}^n$, arbitrary unions of open balls $B_r(x) := \{y \in \mathbb{R}^n : \underbrace{\|x-y\|}_{=\sqrt{\sum_{i=1}^n (x_i-y_i)^2}} < r\}$ can be the open sets τ.

This allows us to define, e.g.:

- convergence (for each $\underbrace{\text{neighborhood } U \text{ of } x}_{\text{any open set that contains } x}$ $\exists N \in \mathbb{N}$ s.t. $x_n \in U$ $\forall n \geq N$)
- continuity (preimages of open sets are open)
- compactness (every open cover has a finite subcover)
- connectedness

However, general topological spaces do not necessarily quantify "closeness".

Next, we will introduce $\underbrace{\text{metric}}_{\text{"distance"}}$, $\underbrace{\text{normed}}_{\text{"length"}}$, and $\underbrace{\text{inner product}}_{\text{"angles"}}$ spaces.

These will give us a notion of:

We continue our discussion of structures that define "closeness" of two elements in a set:

- **Metric spaces** (M, d) have a distance function $d: M \times M \to [0,\infty)$ called a metric.

  A fct. $d: M \times M \to [0,\infty)$ is called a **metric** if:
  - $d(x,y) = 0 \iff x = y$ (definiteness),
  - $d(x,y) = d(y,x)$ $\forall x,y \in M$ (symmetry),
  - $d(x,y) \leq d(x,z) + d(z,y)$ $\forall x,y,z \in M$ (triangle inequality).

  Let us define the open ball around x with radius r as $B_r(x) := \{y \in M : d(x,y) < r\}$.

  Then arbitrary unions of open balls can be def. as the open sets of a topological space (they induce the "metric topology").

  (Every metric space defines a topological space, but not every topological space is induced by a metric.)

- More concretely: Any vector space V over $\mathbb{R}$ (or $\mathbb{C}$ or any other field) together with a **norm** $\|\cdot\|: V \to [0,\infty)$, $x \mapsto \|x\|$ is called **normed space**. A norm is def. by:
  - $\|x\| = 0 \iff x = 0$ (definiteness),
  - $\|\lambda x\| = |\lambda| \, \|x\|$ $\forall \lambda \in \mathbb{R}$ (or $\mathbb{C}$), $x \in V$ (homogeneity),
  - $\|x+y\| \leq \|x\| + \|y\|$ $\forall x,y \in V$ (triangle inequality).

  Any norm defines a metric via $d(x,y) = \|x-y\|$. (But not every metric comes from a norm.)

- Even more special are inner product spaces. A map $\langle \cdot, \cdot \rangle : V \times V \to \mathbb{C}$, $(x,y) \mapsto \langle x,y \rangle$ is called inner product if:
  - $\langle x, \alpha y + \beta z \rangle = \alpha \langle x,y \rangle + \beta \langle x,z \rangle \quad \forall x,y,z \in V, \alpha, \beta \in \mathbb{C}$ (linearity),
  - $\langle x,y \rangle = \overline{\langle y,x \rangle} \quad \forall x,y \in V$ (conjugate symmetry),
  - $\langle x,x \rangle \geq 0$, and $\langle x,x \rangle = 0 \Leftrightarrow x = 0$ (positive definitness).

  Any inner product defines a norm via $\|x\| = \sqrt{\langle x,x \rangle}$. (But not every norm comes from an inner product.)

  Note: An important inequality is Cauchy-Schwarz: $|\langle x,y \rangle| \leq \|x\| \, \|y\| \quad \forall x,y \in V$.

  (see HW 3)

Conceptually:

- topologies define "closeness",
- metrics define "distance",
- norms define "length",
- inner products define "angles".