We continue exploring differentiability in $\mathbb{R}^n$.

<u>Definition:</u> Let $U \subset \mathbb{R}^n$ be open and $f: U \to \mathbb{R}^m$. Then f is called differentiable at $\tilde{x} \in U$ if there is a linear map $A: \mathbb{R}^n \to \mathbb{R}^m$ s.t.

$$\underbrace{\boxed{f(\tilde{x}+h) = f(\tilde{x}) + Ah + r_{\tilde{x}}(h) \qquad \text{with} \quad \lim_{h\to 0} \frac{\|r_{\tilde{x}}(h)\|}{\|h\|} = 0.}}$$

In other words: $\displaystyle\lim_{h\to 0} \frac{\|f(\tilde{x}+h) - f(\tilde{x}) - Ah\|}{\|h\|} = 0.$

We call $A \equiv Df|_{\tilde{x}} \equiv f'(\tilde{x})$ the **total derivative** of f at $\tilde{x}$.

If f is differentiable for all $\tilde{x} \in U$, we say f is differentiable in U.

Note: Clearly differentiability at $\tilde{x} \in U$ implies continuity at $\tilde{x}$ since $\frac{\|r_{\tilde{x}}(h)\|}{\|h\|} \to 0$ implies $\|r_{\tilde{x}}(h)\| \to 0$.

<u>Lemma:</u> If $f: U \to \mathbb{R}^m$ ($U \subset \mathbb{R}^n$ open) is differentiable at $\tilde{x} \in U$, then the derivative $Df|_{\tilde{x}}$ is unique.

<u>Proof:</u> Suppose both A_1 and A_2 are derivatives. Then $B := A_1 - A_2$ satisfies

$$\frac{\|Bh\|}{\|h\|} = \frac{1}{\|h\|} \left\| f(\tilde{x}+h) - f(\tilde{x}) - r_{1,\tilde{x}}(h) - \left(f(\tilde{x}+h) - f(\tilde{x}) - r_{2,\tilde{x}}(h) \right) \right\|$$

$$\leq \frac{\|r_{1,\tilde{x}}(h)\|}{\|h\|} + \frac{\|r_{2,\tilde{x}}(h)\|}{\|h\|} \xrightarrow{h\to 0} 0.$$

Now fix any $u \in \mathbb{R}^n$, $u \neq 0$ and choose $h = tu$, $t \in \mathbb{R}$. Then:

$$0 \xleftarrow{t\to 0} \frac{\|(A_2-A_1)h\|}{\|h\|} = \frac{\|(A_1-A_2)tu\|}{\|tu\|} = \frac{\|(A_1-A_2)u\|}{\|u\|}, \text{ i.e., } A_1u = A_2u \ \forall u \in \mathbb{R}^n$$

$$\Rightarrow A_1 = A_2. \qquad \square$$

<u>Ex.:</u> $f(x_1,x_2) = \begin{pmatrix} x_1^2 + x_1x_2 \\ 2x_1 - x_2^2 \end{pmatrix}$

$$f(x_1+h_1, x_2+h_2) = \begin{pmatrix} (x_1+h_1)^2 + (x_1+h_1)(x_2+h_2) \\ 2(x_1+h_1) - (x_2+h_2)^2 \end{pmatrix} = \underbrace{\begin{pmatrix} x_1^2 + x_1x_2 \\ 2x_1 - x_2^2 \end{pmatrix}}_{=f(x_1,x_2)} + \underbrace{\begin{pmatrix} 2x_1h_1 + x_1h_2 + x_2h_1 \\ 2h_1 - 2h_2x_2 \end{pmatrix}}_{= \underbrace{\begin{pmatrix} 2x_1+x_2 & x_1 \\ 2 & -2x_2 \end{pmatrix}}_{Df|_x} \begin{pmatrix} h_1 \\ h_2 \end{pmatrix}} + \underbrace{\begin{pmatrix} h_1^2 + h_1h_2 \\ -h_2^2 \end{pmatrix}}_{=r(h)}$$

with $\frac{\|r_h\|^2}{\|h\|^2} = \frac{h_1^4 + h_1^2h_2^2 + h_2^4}{h_1^2+h_2^2} \xrightarrow{h_1,h_2\to 0} 0.$

Next we consider derivatives in different directions:

<u>Definition:</u> $f: U \to \mathbb{R}^m$ ($U \subset \mathbb{R}^n$ open) is differentiable at $\tilde{x} \in U$ in the direction $u \in \mathbb{R}^n$, $\|u\| = 1$, if

$$\lim_{t\to 0} \frac{f(\tilde{x}+tu) - f(\tilde{x})}{t}$$ exists. Then this limit is denoted by $D_u f|_{\tilde{x}}$ and called $\boxed{\text{directional derivative}}$. (or derivative in direction u)

If f is differentiable in the direction e_j, we call $D_{e_j} f|_{\tilde{x}} = \frac{\partial f}{\partial x_j}(\tilde{x})$ the j-th $\boxed{\text{partial derivative}}$ of f at $\tilde{x}$.

In other words: $$\frac{\partial f_k}{\partial x_j}(\tilde{x}) = \lim_{t\to 0} \frac{f_k(\tilde{x}_1, \dots, \tilde{x}_{j-1}, \tilde{x}_j + t, \tilde{x}_{j+1}, \dots, \tilde{x}_n) - f_k(\tilde{x})}{t}.$$

the 1-dimensional derivative of f_k in the variable x_j only (keeping all other variables fixed)

<u>Ex.:</u> $f(x_1,x_2) = \begin{pmatrix} x_1^2 + x_1x_2 \\ 2x_1 - x_2^2 \end{pmatrix} \Rightarrow \frac{\partial f}{\partial x_1} = \begin{pmatrix} 2x_1 + x_2 \\ 2 \end{pmatrix}, \ \frac{\partial f}{\partial x_2} = \begin{pmatrix} x_1 \\ -2x_2 \end{pmatrix}$

Note that in the example we have $Df|_x = \left(\frac{\partial f}{\partial x_1}, \frac{\partial f}{\partial x_2}\right)$.

The first result is:

Theorem: If $f: U \to \mathbb{R}^m$ ($U \subset \mathbb{R}^n$ open) is differentiable at $\tilde{x} \in U$, then all directional derivatives at $\tilde{x}$ exist. In this case, the derivative in direction $u \in \mathbb{R}^n$, $\|u\| = 1$, is given by $D_u f|_{\tilde{x}} = \underbrace{Df|_{\tilde{x}}}_{m \times n \text{ matrix}} \underbrace{u}_{\in \mathbb{R}^n}$.

$$Df = \begin{pmatrix} \frac{\partial f_1}{\partial x_1} & \cdots & \frac{\partial f_1}{\partial x_n} \\ \vdots & & \vdots \\ \frac{\partial f_m}{\partial x_1} & \cdots & \frac{\partial f_m}{\partial x_n} \end{pmatrix}$$ "Jacobian matrix"

In particular, $\underbrace{\frac{\partial f_i(\tilde{x})}{\partial x_j}}_{\substack{\text{derivative of the} \\ i\text{-th component of } f \\ \text{w.r.t. } x_j}} = \underbrace{\left(Df|_{\tilde{x}}\right)_{ij}}_{\substack{(i,j) \text{ matrix entry of} \\ \text{the total derivative,} \\ = \text{the matrix of this} \\ \text{linear map in the basis } (e_j)}}$.

Proof: f differentiable at $\tilde{x}$ means $\lim_{h \to 0} \frac{\|f(\tilde{x}+h) - f(\tilde{x}) - Df \cdot h\|}{\|h\|} = 0$.

In particular, for $u \in \mathbb{R}^n$, $\|u\| = 1$, we can choose $h = tu$ and get

$$0 = \lim_{t \to 0} \frac{\|f(\tilde{x}+tu) - f(\tilde{x}) - Df \cdot u t\|}{t} = \lim_{t \to 0} \left\| \frac{f(\tilde{x}+tu) - f(\tilde{x})}{t} - Df \cdot u \right\|, \text{ i.e.,}$$

$$\lim_{t \to 0} \frac{f(\tilde{x}+tu) - f(\tilde{x})}{t} = Df \cdot u. \quad \square$$

But: There are examples of functions where all partial derivatives exist, but that are not differentiable (total derivative does not exist). See Homework.