Last time, we introduced the Hessian H_f of a function $f: U \to \mathbb{R}$ ($U \subset \mathbb{R}^n$ open) as

$\boxed{(H_f)_{ij} := \frac{\partial^2 f}{\partial x_i \partial x_j}}$. For $f \in C^2$, we found $(H_f)_{ij} = (H_f)_{ji}$ (Clairaut, Schwarz).

Similar to functions in $\mathbb{R}$, we can do a Taylor expansion. Let us write it down here up to second order (and for $f: \mathbb{R}^n \to \mathbb{R}$ only).

Theorem (Taylor, 2nd order): Let $f: U \to \mathbb{R}$, $U \subset \mathbb{R}^n$ open, $f \in C^2(U)$. Let $x \in U$ and $h \in \mathbb{R}^n$ such that $x + th \in U$ $\forall t \in [0,1]$. Then

$$f(x+h) = f(x) + Df|_x h + \frac{1}{2} \underbrace{\langle h, H_f(x) h \rangle}_{= h^T H_f(x) \cdot h} + r_x(h), \text{ with } \frac{\|r_x(h)\|}{\|h\|^2} \xrightarrow{h \to 0} 0.$$

Proof: Follows from applying 1-d Taylor to $g(t) := f(p + th)$. $\square$

The Hessian can be used to determine whether an extremum is a maximum or minimum:

Theorem: Let $f: U \to \mathbb{R}$, $U \subset \mathbb{R}^n$ open, $f \in C^2(U)$, with $(\nabla f)(x) = 0$ for some $x \in U$. Then:

- If $H_f(x)$ is positive definite (i.e., $\langle h, H_f(x) h \rangle > 0$ $\forall h \in \mathbb{R}^n, h \neq 0$), then f has a $\boxed{\text{local minimum}}$ at x.
- If $H_f(x)$ is negative definite (i.e., $\langle h, H_f(x) h \rangle < 0$ $\forall h \in \mathbb{R}^n, h \neq 0$), then f has a $\boxed{\text{local maximum}}$ at x.

Proof: Follows from the Taylor expansion (making h very small). $\square$

Note: Since H is symmetric, all eigenvalues are real. Then H is positive definite if and only if all eigenvalues are positive.

A very simple example: $f(x,y) = -x^2 - y^2$.

$$\nabla f = \begin{pmatrix} \frac{\partial f}{\partial x} \\ \frac{\partial f}{\partial y} \end{pmatrix} = \begin{pmatrix} -2x \\ -2y \end{pmatrix}. \quad \nabla f = 0 \text{ for } (x,y) = (0,0).$$

$$H_f((x,y)=(0,0)) = \begin{pmatrix} -2 & 0 \\ 0 & -2 \end{pmatrix}, \text{ so } H_f(0) \text{ is negative definite.}$$

$\Rightarrow f$ has a maximum at $(0,0)$.

Other examples (see geogebra pictures below):

- $f(x,y) = x^2 - y^2 + 2 \Rightarrow H_f(0,0) = \begin{pmatrix} 2 & 0 \\ 0 & -2 \end{pmatrix} \Rightarrow$ saddle point ($\langle h, H_f h \rangle > 0$ for some h, and $\langle h, H_f h \rangle < 0$ for others)

- $f(x,y) = x^3 - y^2 + 2 \Rightarrow H_f(0,0) = \begin{pmatrix} 0 & 0 \\ 0 & -2 \end{pmatrix} \Rightarrow$ degenerate point ($\langle h, H_f h \rangle = 0$ for some $h \in \mathbb{R}^n$)

- $f(x,y) = y^3 - 3x^2 y + 2 \Rightarrow H_f(0,0) = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \Rightarrow$ degenerate point ("monkey saddle", see pictures)

2.3 The Inverse and Implicit Function Theorems

Question: Under which conditions is $\underbrace{f: U \to \mathbb{R}^n \quad (U \subset \mathbb{R}^n \text{ open})}$ invertible?

Here f goes from a subset of $\mathbb{R}^n$ into a subset of $\mathbb{R}^n$.

And: If f is invertible and differentiable, is then $\underbrace{f^{-1}}_{\text{the inverse of } f}$ also differentiable?

Reminder:

- $f: X \to Y$ is called $\boxed{\text{injective}}$ (or "one-to-one") if $f(x_1) = f(x_2)$ implies $x_1 = x_2$.
  (In other words: Given $y \in Y$, then $f(x) = y$ for at most one $x \in X$.)

  Ex.: $f: \mathbb{R} \to \mathbb{R}, x \mapsto e^x$ is injective, but $f: \mathbb{R} \to \mathbb{R}, x \mapsto x^2$ is not.

- $f: X \to Y$ is called $\boxed{\text{surjective}}$ (or "onto") if $\forall y \in Y$ there is an $x \in X$ s.t. $f(x) = y$.

  Ex.: $f: \mathbb{R} \to \mathbb{R}, x \mapsto e^x$ is not surjective, but $f: \mathbb{R} \to (0, \infty), x \mapsto e^x$ is.

- $f: X \to Y$ is called $\boxed{\text{bijective}}$ if it is injective and surjective.
  If $f: X \to Y$ is bijective, then it has an inverse $f^{-1}: Y \to X$.
  (i.e., $f(f^{-1}(y)) = y \ \forall y \in Y$, or $f^{-1}(f(x)) = x \ \forall x \in X$.)

From Analysis and Calculus we know the case $n = 1$:

If f is continuously differentiable and $f'(p) \neq 0$, then f is invertible in a neighborhood of p, f^{-1} is continuously differentiable, and $\underbrace{(f^{-1})'(f(p)) = \frac{1}{f'(p)}}$.

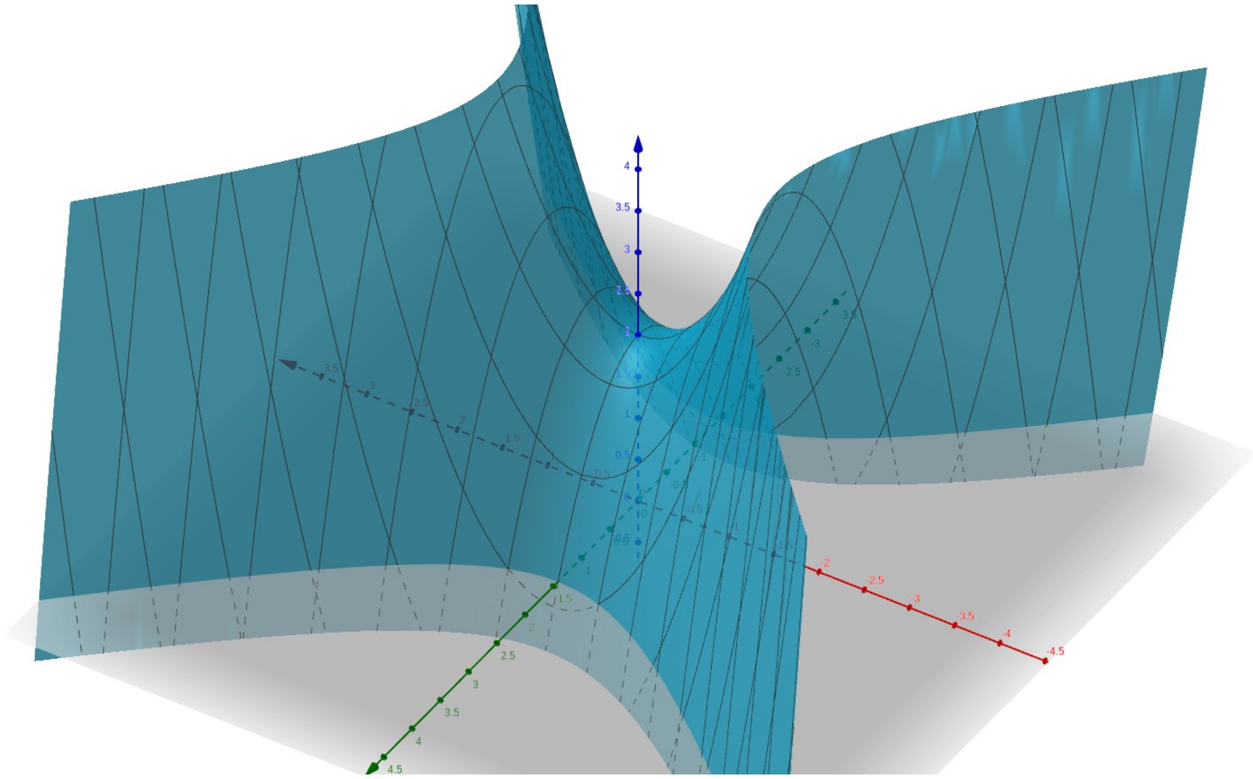
If $f(p) = q$, then $(f^{-1})'(q) = \dfrac{1}{f'(f^{-1}(q))}$

For $n > 1$: Use linear approximation of f; then f should be invertible (in small neighborhood) if Df is invertible.

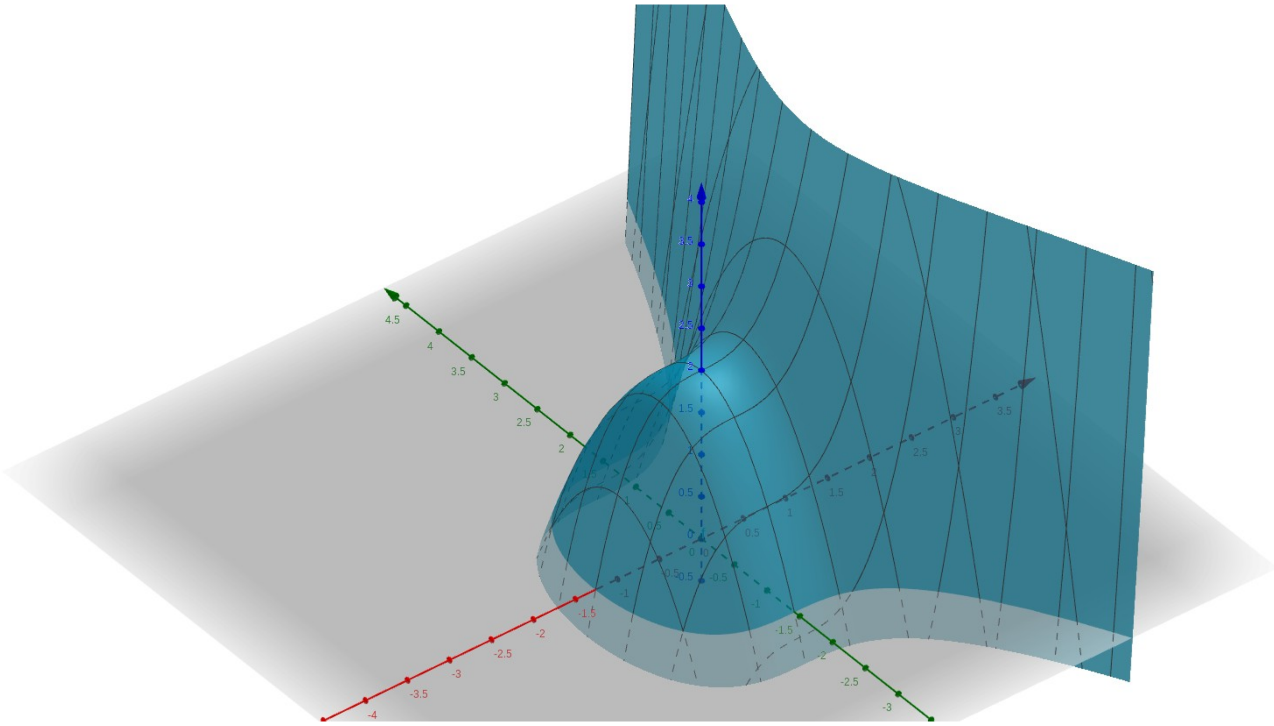
Precise statement: next time.

The following pictures were generated with https://www.geogebra.org/3d.

$$f(x,y)=x^2-y^2+2$$



$$f(x,y)=x^3-y^2+2$$



$$f(x,y)=y^3-3x^2y+2$$

("Monkey Saddle")