3. Integrals

Generally, there are at least 3 ways to integrate in many variables:

- Successive 1-dim. Riemann integrals: $\int\limits_{[a,b]\times[c,d]} f(x_1,x_2)\,d^2x := \int_c^d \left(\int_a^b f(x_1,x_2)\,dx_1 \right) dx_2.$

  Then an important question is: Is $\int \left(\int f(x_1,x_2)dx_1 \right) dx_2 = \int \left(\int f(x_1,x_2) dx_2 \right) dx_1$?

- Re-define the Riemann integral in n-dim., using partitions of $\mathbb{R}^n$.

  Question: Is it equal to successive 1-dim. integration?

- Lebesgue integral: see Analysis III.

3.1 Partial Integrals

We first consider partial integrals, i.e., $F(y) := \int_a^b f(x,y)\,dx$.

Here, $I = [a,b] \times [\alpha,\beta] = I_1 \times I_2$, $f: I \to \mathbb{R}$, $\underbrace{f(\cdot,y)}_{}$ is integrable for every $y \in I_2$, $F: I_2 \to \mathbb{R}$.

f as a fct. of the variable in the first slot, for fixed y

A key property to understand partial integrals is uniform continuity.

Definition: Let X, Y be metric spaces. Then $f: X \to Y$ is called **uniformly continuous** if $\underbrace{\forall \varepsilon > 0 \ \exists \delta > 0 \text{ s.t. } \forall x, x' \in X}_{\text{One } \varepsilon \text{ works } \forall x, x' \in X \text{ (compare with uniform convergence).}}$ with $d(x,x') < \delta$ we have $d(f(x), f(x')) < \varepsilon$.

We will use the following result (applied to the compact sets I, I_1, I_2):

Theorem: If K is compact and $f: K \to Y$ continuous, then f is uniformly continuous.

Proof: Let $\varepsilon > 0$. For each $x \in K$, f is continuous, i.e., $\exists \delta_x > 0$ s.t. $\forall y$ with $\underbrace{d(x,y) < \delta_x}_{\Leftrightarrow y \in B_{\delta_x}(x)}$ we have $d(f(x), f(y)) < \frac{\varepsilon}{2}$.

Now $\{B_{\delta_x/2}(x)\}_{x \in K}$ is an open cover of K, and since K compact there exists a finite subcover $\{B_{\delta_{x_i}/2}(x_i)\}_{i=1,\dots,n}$.

We choose $\delta := \min\limits_{i=1,\dots,n} \frac{\delta_{x_i}}{2}$. Then for all $x, y \in K$ with $d(x,y) < \delta$ we have:

- $d(x, x_j) < \frac{\delta_{x_j}}{2}$ for some $j = 1, \dots, n$ (since $\{B_{\delta_{x_i}/2}(x_i)\}_{i=1,\dots,n}$ is an open cover)
- $d(y, x_j) \leq d(y,x) + d(x, x_j) \leq \delta + \frac{\delta_{x_j}}{2} \leq \delta_{x_j}$

Thus: $d(f(x), f(y)) \leq \underbrace{d(f(x), f(x_j))}_{< \frac{\varepsilon}{2}} + \underbrace{d(f(y), f(x_j))}_{< \frac{\varepsilon}{2}} < \varepsilon$. $\square$

(Note: We could also use sequential compactness for the proof.)

Back to the partial integral $F(y) := \int_a^b f(x,y)\,dx$.

First, we aim at proving $\frac{dF(y)}{dy} = \int_a^b \frac{\partial f(x,y)}{\partial y}\, dx$. A first (intermediate) result is:

Theorem: If $\underbrace{f \in C(I)}_{f \text{ is continuous on } I = I_1 \times I_2}$, then $F \in C(I_2)$.

Proof: Let $\varepsilon > 0$. $f \in C(I) \Rightarrow f$ uniformly continuous on $I \Rightarrow \exists\, \delta > 0$ s.t.

$\forall x, y, y'$ with $\underbrace{|y - y'|}_{= |(x,y) - (x,y')|} < \delta$: $|f(x,y) - f(x,y')| < \frac{\varepsilon}{b-a}$.

Then $|F(y) - F(y')| = \left| \int_a^b (f(x,y) - f(x,y'))\, dx \right| \leq \int_a^b \underbrace{|f(x,y) - f(x,y')|}_{\leq \frac{\varepsilon}{b-a}}\, dx \leq (b-a) \frac{\varepsilon}{b-a} = \varepsilon.$ $\square$

Then the following holds:

Theorem (Leibniz rule I):
If $f \in C(I)$ and $\frac{\partial f}{\partial y} \in C(I)$, then $F \in C^1(I_2)$ and $\underbrace{\frac{dF}{dy}(y) = \int_a^b \frac{\partial f}{\partial y}(x,y)\, dx}_{(*)}$.

Proof: Let $\varepsilon > 0$. Since $\frac{\partial f}{\partial y} \in C(I)$, it is uniformly continuous.

Therefore, $\exists\, \delta > 0$ s.t. $\forall x \in I_1, y, y' \in I_2$ with $|y - y'| < \delta$:

$$\left| \frac{\partial f}{\partial y}(x,y) - \frac{\partial f}{\partial y}(x,y') \right| < \frac{\varepsilon}{b-a}.$$

Then, for $|h| < \delta$ $(y + h \in I_2)$:

$$\Rightarrow \left| \frac{F(y+h) - F(y)}{h} - \int_a^b \frac{\partial f}{\partial y}(x,y)\, dx \right| = \left| \int_a^b \left(\underbrace{\frac{f(x,y+h) - f(x,y)}{h}}_{= \frac{\partial f}{\partial y}(x, y+\theta h),\ 0<\theta<1, \text{ by the mean-value thm.}} - \frac{\partial f}{\partial y}(x,y) \right) dx \right|$$

$$< (b-a) \frac{\varepsilon}{b-a} = \varepsilon.$$

$\Rightarrow F$ differentiable and $(*)$ holds. With previous thm. applied to $\frac{\partial f}{\partial y}$, F' is continuous (i.e., $F \in C^1(I_2)$.) $\square$

What about indefinite integrals?

Theorem (Leibniz rule II): Let $I_1 = [a, \infty)$, $I_2 = [\alpha, \beta]$, $I = I_1 \times I_2$, f and $\frac{\partial f}{\partial y} \in C(I)$.

Assume: (i) $F(y) = \int_a^\infty f(x,y)\,dx$ converges $\forall y \in I_2$.

(ii) $\int_a^\infty \frac{\partial f}{\partial y}(x,y)\,dx$ converges absolutely and uniformly on I_2.

↳ i.e., $g_n(y) := \int_a^n \frac{\partial f}{\partial y}(x,y)\,dx$ conv. abs. and uniformly

Then $F \in C^1(I_2)$ and $F'(y) = \int_a^\infty \frac{\partial f}{\partial y}(x,y)\,dx$.

Proof: Compared to the previous thm., we cannot use anymore that $\frac{\partial f}{\partial y}$ is uniformly continuous.

But we find, for any $b > a$:

$$\left| \frac{F(y+h) - F(y)}{h} - \int_a^\infty \frac{\partial f}{\partial y}(x,y)\,dx \right| \le \int_a^\infty \left| \frac{\partial f}{\partial y}(x, y+\theta h) - \frac{\partial f}{\partial y}(x,y) \right| dx$$

$$\le \underbrace{\int_a^b \left| \frac{\partial f}{\partial y}(x, y+\theta h) - \frac{\partial f}{\partial y}(x,y) \right| dx}_{< \frac{\varepsilon}{3}} + \underbrace{\int_b^\infty \left| \frac{\partial f}{\partial y}(x, y+\theta h) \right| dx}_{< \frac{\varepsilon}{3}} + \underbrace{\int_b^\infty \left| \frac{\partial f}{\partial y}(x,y) \right| dx}_{< \frac{\varepsilon}{3}} < \varepsilon$$

bc. as in previous proof: we can choose $\delta = \frac{\varepsilon}{3(b-a)}$ due to uniform continuity

by uniform convergence
$\left(\forall \varepsilon > 0\ \exists N \in \mathbb{N} \text{ s.t. } \forall n \ge N\ \forall y \in I_2 : \int_b^n \left| \frac{\partial f}{\partial y} \right| dx < \frac{\varepsilon}{3} \right)$

Continuity of $F'(y) = \int_a^\infty \frac{\partial f}{\partial y}(x,y)\,dx$ follows as before and with the same argument of splitting $\int_a^\infty \ldots = \int_a^b \ldots$. $\square$