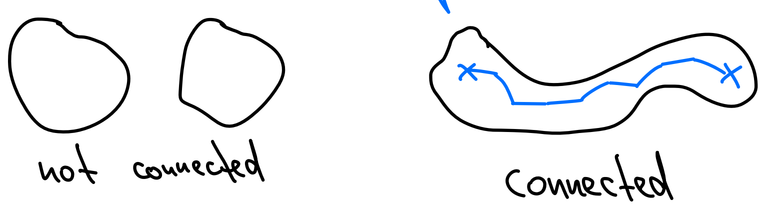
3.2 The Riemann Integral in $\mathbb{R}^n$

Strategy: We define a Riemann integral "from scratch" for $f: \overline{D} \to \mathbb{R}$, with $\overline{D} \subset \mathbb{R}^n$ a "closed domain with content". Afterwards we make the connection to repeated 1-dim. Riemann integrals.

To define "volume", we first need a few topological notions:

- An open set $A \subset \mathbb{R}^n$ is **connected** if and only if any two points in A can be connected by a polygonal path. (Note: there is a more general topological def. of connectedness.)

  One can show that this is equivalent to taking any continuous path.



Definition: A **domain** in $\mathbb{R}^n$ is a non-empty connected open set.

- We call x a **boundary point** of $A \subset \mathbb{R}^n$ if every open neighborhood of x contains a point in A and in A^c. ($A^c = \mathbb{R}^n \setminus A$ is the complement of A.)

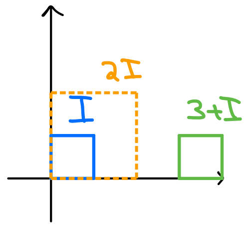
- We denote:
  - $\partial A = \{\text{all boundary points of } A\}$ the **boundary** of A (e.g., $\partial\{\|x\|<r\} = \{\|x\|=r\}$)
  - $\overline{A} = A \cup \partial A$ the **closure** of A (e.g., $\overline{(0,1)} = (0,1) \cup \{0\} \cup \{1\} = [0,1]$)
  - If $D \subset \mathbb{R}^n$ is a domain, we call $\overline{D}$ **closed domain**

We now aim at defining the content or "volume" $S(A)$ for $A\subset\mathbb{R}^n$.

We define:

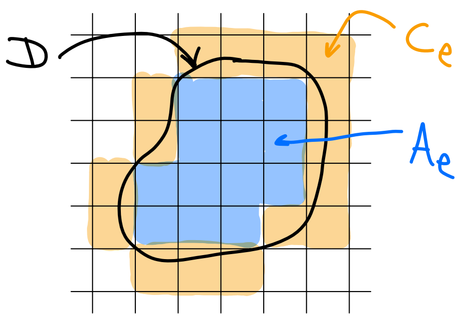
- The unit cell $I=[0,1]^n$ has content $S(I)=1$.
- Let $I_k=k+I$ be the unit cell translated by k, ρI the unit cell dilated by ρ $(\rho>0)$.

  Then $S(k+\rho I)=\rho^n S(I)$.



Then, for $\rho>0$, we divide $\mathbb{R}^n$ into cells, $\mathbb{R}^n=\bigcup_{k\in\mathbb{Z}^n}\rho I_k$, and we def. for bounded closed domains $\bar{D}\subset\mathbb{R}^n$:

- $A_\rho=\bigcup\{\rho I_k \text{ inside } D\}$
- $C_\rho=\bigcup\{\rho I_k \text{ hits the boundary of } D\}$



<u>Definition:</u> Given a domain $D\subset\mathbb{R}^n$, we say D (and $\bar{D}$) "has content" or "is Jordan measurable" if $\lim_{\rho\to0}S(A_\rho)$ and $\lim_{\rho\to0}S(A_\rho\cup C_\rho)$ exist and are equal. The result is called "Jordan content" or "Jordan measure" $S(D)=S(\bar{D})=\lim_{\rho\to0}S(A_\rho)=\lim_{\rho\to0}S(A_\rho\cup C_\rho)$.

Informally: D is Jordan measurable if its boundary is not too large/wild.

<u>Example:</u> $D=[0,1]\cap\mathbb{Q}$. (Note: Not a domain, but we can still use the def. above.)

Since $\partial\mathbb{Q}=\mathbb{R}$, we have $S(A_\rho)=0$, $S(A_\rho\cup C_\rho)=S(C_\rho)=1$, so D is not Jordan measurable. (Note: D will turn out to be Lebesgue measurable.)

Next: Partitions, Riemann sums $\to$ Riemann integrability

For the following definitions, $\overline{D} \subset \mathbb{R}^n$ is a bounded closed domain with content.

Definition: A **partition** of $\overline{D}$ is a family $T = \{\overline{D}_j, j = 1, \dots, k\}$ such that

a) $\overline{D}_j \subset \overline{D}$ are closed sub-domains with content,

b) $\{D_j\}$ disjoint,

c) $\overline{D} = \bigcup_{j=1}^{k} \overline{D}_j$.

We call $\lambda(T)$ = the maximal diameter of all $\overline{D}_j$'s the "parameter" or "**mesh**" of T.

Definition: Let $f: \overline{D} \to \mathbb{R}$ be bounded. A **Riemann sum** for f is a sum

$$\sigma(f, T, x_1, \dots, x_k) = \sum_{j=1}^{k} f(x_j) \underbrace{S(\overline{D}_j)}, \text{ with } x_j \in \overline{D}_j.$$

note: in 1-dim.: $S(\overline{D}_j) = \Delta x_j = x_j - x_{j-1}$

With that we can define:

Definition: $f: \overline{D} \to \mathbb{R}$ bounded is **Riemann integrable** on D (or $\overline{D}$) if $\exists I \in \mathbb{R}$ s.t.:

$$\left.\begin{array}{l} \forall \varepsilon > 0 \ \exists \delta > 0 \text{ s.t. } \forall \text{ partitions } T \text{ with } \lambda(T) < \delta \text{ and } \forall x_j \in D \text{ we have} \\ \qquad |\sigma(f, T, x_1, \dots, x_k) - I| < \varepsilon. \end{array}\right\} (*)$$

In this case we write $I = \int_{\overline{D}} f \, dS$, and $f \in \overbrace{R(D)}^{\text{Riemann integrable on } D}$.

Note: $(*)$ can be expressed as $\lim_{\lambda(T) \to 0} \sigma(f, T) = I$.

Note: We could as well define upper and lower Riemann integrals and call fct.s Riemann integrable if both coincide.

$$\hookrightarrow \underline{\sigma}(f,T) = \sum_j \underbrace{\left(\inf_{x\in D_j} f(x)\right)}_{=:m_j} S(\overline{D}_j) \quad , \quad \overline{\sigma}(f,T) = \sum_j \underbrace{\left(\sup_{x\in D_j} f(x)\right)}_{=:M_j} S(\overline{D}_j)$$

$$\Rightarrow \overline{\int_D} f\,dS := \inf_T \overline{\sigma}(f,T)$$

$$\underline{\int_D} f\,dS := \sup_T \underline{\sigma}(f,T)$$

$$\Rightarrow \text{If } \overline{\int_D} f\,dS = \underline{\int_D} f\,dS \text{, then } f\in R(D).$$

Theorem (Riemann criterion):

$f\in R(D) \iff \forall \varepsilon>0\ \exists \delta>0$ s.t. $\sum_j \left|M_j - m_j\right| S(\overline{D}_j) < \varepsilon \ \forall T$ with $\lambda(T) < \delta$.

Proof: omitted.