

Next: Properties of the Riemann integral and connection to repeated 1-dim. integrals.

Recall: $R(D)$ = set of Riemann integrable functions on a domain D .

Theorem:

(i) $R(D)$ is a vector space, \int is linear.

(ii) $R(D)$ is an algebra containing 1, and $\int_D 1 dS = S(D)$.

i.e., if $f, g \in R(D)$, then $fg \in R(D)$

(iii) \int is monotonic and $m S(D) \leq \int_D f dS \leq M S(D)$, $m = \inf_{x \in D} f(x)$, $M = \sup_{x \in D} f(x)$.

i.e., if $f \leq g$, then $\int_D f dS \leq \int_D g dS$

(iv) $C(\bar{D}) \subset R(D)$ (Continuous fct.s are Riemann integrable.)

(v) A mean-value theorem (MVT) holds:

If $f \in C(\bar{D})$, then $\exists p \in \bar{D}$ s.t. $\int_D f dS = f(p) S(D)$.

(vi) If $\{D_j\}$ is a partition of D and $f \in R(D)$, then $f \in R(D_j)$ and

$$\int_D f dS = \sum_j \int_{D_j} f dS.$$

$$\int_D g dS = \int_{\cup D_j} g dS$$

(vii) If $f, g \in R(D)$ and $f = g$ on D , then $\int_D f dS = \int_D g dS = \int_D g dS$.

(viii) If $f \in R(D)$, $g \in C([m, M])$, then $g \circ f \in R(D)$.

(ix) If $f \in R(D)$, then $|f| \in R(D)$ and $|\int_D f dS| \leq \int_D |f| dS$.

These properties have nice short proofs, see Kantorovitz. Here, let us just give a

Proof of (ii): Let $f \in R(D)$. We prove that then $f^2 \in R(D)$. This implies:

$$\text{If } f, g \in R(D) \Rightarrow (f+g)^2, (f-g)^2 \in R(D) \Rightarrow (f+g)^2 - (f-g)^2 = 4fg \in R(D) \\ \Rightarrow fg \in R(D).$$

(left to prove: $f^2 \in R(D)$). Let $\epsilon > 0$. Then $\exists \delta > 0$ s.t. $\forall T$ with $\lambda(T) < \delta$ we have:

$$\sum_j (M_j - m_j) S(D_j) < \frac{\epsilon}{2M} \quad (\text{Riemann criterion}), \text{ with } M = \sup_j M_j.$$

$$\Rightarrow \sum_j \underbrace{(M_j^2 - m_j^2)}_{=(M_j - m_j)(M_j + m_j) \leq 2M(M_j - m_j)} S(D_j) \leq 2M \sum_j (M_j - m_j) S(D_j) \leq 2M \frac{\epsilon}{2M} = \epsilon \Rightarrow f^2 \in R(D). \quad \square$$

↑ extra proof not discussed in class

The connection to partial integrals is:

Theorem: For $I = [a, b] \times [\alpha, \beta]$, let $f \in R(I)$ and $f(\cdot, y) \in R([a, b])$ for each $y \in [\alpha, \beta]$.

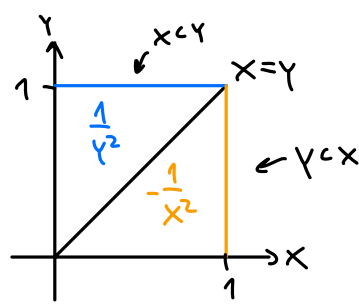
$$\text{Then } F(y) := \int_a^b f(x, y) dx \in R([\alpha, \beta]) \text{ and } \int_I f dS = \int_{\alpha}^{\beta} \int_a^b f(x, y) dx dy.$$

Proof: See Kantorovitz

Note:

- If $f \notin C(I)$, then order of integration may matter, and even if the iterated integrals exist, f is not necessarily Riemann integrable.
- Just $f \in R(I)$ does not necessarily imply that iterated 1-dim. integrals exist.

Example: let $f(x,y) := \begin{cases} \frac{1}{y^2} & \text{for } 0 < x < y < 1 \\ -\frac{1}{x^2} & \text{for } 0 < y < x < 1 \end{cases}$



• For fixed $y > 0$: $\int_0^1 f(x,y) dx = \int_0^y \frac{1}{y^2} dx + \int_y^1 (-\frac{1}{x^2}) dx = \frac{y}{y^2} + \frac{1}{x} \Big|_y^1 = \frac{1}{y} + 1 - \frac{1}{y} = 1$.

• For fixed $x > 0$: $\int_0^1 f(x,y) dy = \int_0^x (-\frac{1}{x^2}) dy + \int_x^1 \frac{1}{y^2} dy = -\frac{x}{x^2} - \frac{1}{y} \Big|_x^1 = -\frac{1}{x} - 1 + \frac{1}{x} = -1$.

$\Rightarrow \int_0^1 \int_0^1 f(x,y) dx dy = 1 \neq -1 = \int_0^1 \int_0^1 f(x,y) dy dx$

$\Rightarrow f(\cdot, y), f(x, \cdot) \in R([0,1])$, so according to thm. 1, $f \notin R([0,1]^2)$

Note:

• With the previous thm. on exchanging order of integration we get:

$f \in C(I) \Rightarrow \int_I f dS = \int_a^b \int_a^b f(x,y) dx dy = \int_a^b \int_a^b f(x,y) dy dx$. (In example above: $f \notin C(I)$.)

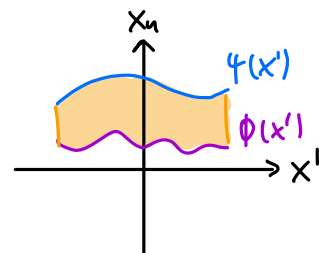
• Another useful theorem is "Fubini-Tonelli": If $\int_a^b \int_a^b |f(x,y)| dx dy$ exists and is finite, then order of integration can be interchanged. [Check that also this criterion is violated for the example above.]

Next: Connection of Riemann integral to variable boundaries.

Definition: Let $U \subset \mathbb{R}^{n-1}$ be bounded, let $\phi, \psi: U \rightarrow \mathbb{R}$ be $C(U)$, define $x' = (x_1, \dots, x_{n-1})$.

Then $\bar{D} := \{x \in \mathbb{R}^n: \phi(x') \leq x_n \leq \psi(x'), x' \in U\}$ is called a **normal domain**.

Note: $S(D) = \int_D 1 dS = \int_U \int_{\phi(x')}^{\psi(x')} dx_n dx' = \int_U (\psi(x') - \phi(x')) dx'$.
or: $d^{n-1}x'$, or $dS(x')$



Then:

Theorem: If $f \in C(\bar{D})$, then $\int_D f dS = \int_u^{\varphi(x)} \int_{\phi(x')} f(x', x_n) dx_n dx'$.

Sketch of proof: Let $I \subset \mathbb{R}^n$ be a box with $\bar{D} \subset \text{int}(I)$. We extend f by zero to I , i.e., $f|_{I \setminus \bar{D}} := 0$.

Then $\int_D f dS = \int_I f dS$, which can be proven by choosing a small enough partition.

(See lemma 4.2.7 in Kantorovitz.)

$$\begin{aligned} \Rightarrow \int_D f dS &= \int_I f dS = \int_{I'} \underbrace{\int_{\alpha}^{\beta} f(x', x_n) dx_n}_{\substack{\text{1-dim. Riemann integral} \\ \varphi(x')}} dx' = \int_{I'} \int_{\phi(x')}^{\varphi(x')} f(x', x_n) dx_n dx' \\ &= \int_u^{\varphi(x)} \int_{\phi(x')}^{\varphi(x')} f(x', x_n) dx_n dx' + 0. \quad \square \end{aligned}$$

split $I' = \bar{u} \cup I' \setminus \bar{u}$

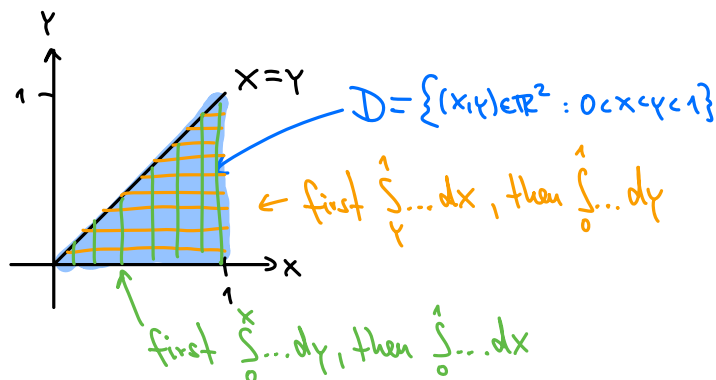
Example: $f(x, y) = ye^{x^3}$.

$$\int_0^1 \int_0^1 ye^{x^3} dx dy$$

by thm. above $\Rightarrow \int_0^1 \int_0^x ye^{x^3} dy dx$

$$= e^{x^3} \int_0^x y dy = e^{x^3} \left. \frac{y^2}{2} \right|_{y=0}^{y=x} = \frac{x^2}{2} e^{x^3}$$

$$= \int_0^1 \frac{x^2}{2} e^{x^3} dx = \int_0^1 \frac{1}{6} \left(\frac{d}{dx} e^{x^3} \right) dx = \frac{1}{6} e^{x^3} \Big|_0^1 = \frac{1}{6} (e-1).$$



The last important thing we need for the Riemann integral in \mathbb{R}^n is the change of variable formula.

Theorem: Let $U, V \subset \mathbb{R}^n$ be domains with content, let $\phi: U \rightarrow V$ be a diffeomorphism (i.e., $\phi \in C^1$, ϕ invertible, and $\phi^{-1} \in C^1$). Then, for $f \in \mathcal{R}(V)$ we have

$$\int_V f \, dx = \int_U f(\phi(u)) |\det D\phi(u)| \, du.$$

$= \int_V f \, ds$ $= \int_U f \circ \phi |\det D\phi| \, ds$

(We skip the proof.)

Note:

- Think of $|\det D\phi(u)| \, du$ as the transformed volume element, keeping in mind that the determinant is a volume!
- In 1-dim. we know the substitution formula

$$\int_a^b f(x) \, dx = \int_\alpha^\beta f(\phi(u)) \phi'(u) \, du \quad , \text{ for } a < b \quad , \quad \phi: [\alpha, \beta] \rightarrow [a, b] \quad , \quad \phi' > 0.$$

Note that when $\phi' < 0$, then $\int_a^b f(x) \, dx = \int_\beta^\alpha f(\phi(u)) \phi'(u) \, du = \int_\alpha^\beta f(\phi(u)) (-\phi'(u)) \, du$,

so indeed $\int_a^b f(x) \, dx = \int_\alpha^\beta f(\phi(u)) |\phi'(u)| \, du$, as in the theorem.