Recall:

Theorem: Let $U, V \subset \mathbb{R}^n$ be domains with content, let $\Phi: U \to V$ be a diffeomorphism (i.e., $\Phi \in C^1$, Φ invertible, and $\Phi^{-1} \in C^1$). Then, for $f \in R(V)$ we have

$$\boxed{\underbrace{\int_V f\, dx}_{= \int_V f\, dS} = \underbrace{\int_U f(\Phi(u))\, |\det D\Phi(u)|\, du}_{= \int_U f \circ \Phi\, |\det D\Phi|\, dS}.}$$

Important examples:

$$\boxed{\text{Polar coordinates: } \Phi(r,\varphi) = \begin{pmatrix} r\cos\varphi \\ r\sin\varphi \end{pmatrix}}$$

$$\Rightarrow D\Phi(r,\varphi) = \begin{pmatrix} \cos\varphi & -r\sin\varphi \\ \sin\varphi & r\cos\varphi \end{pmatrix}, \text{ so } \det(D\Phi(r,\varphi)) = r\cos^2\varphi + r\sin^2\varphi = r.$$

$$\Rightarrow \int_{B_R(0)} f(x)\, dx = \int_0^R \int_0^{2\pi} f(r\cos\varphi, r\sin\varphi)\, d\varphi\, r\, dr.$$

- E.g., area of a circle: $\int_{B_R(0)} 1\, dx = \int_0^R \int_0^{2\pi} 1\, d\varphi\, r dr = 2\pi \int_0^R r\, dr = \pi R^2.$

- E.g., area of an ellipse $E := \{(x,y) \in \mathbb{R}^2 : \frac{x^2}{a^2} + \frac{y^2}{b^2} \leq 1\}$, given $a, b > 0$.

We can def. $\Phi: B_1(0) \to E$, $\Phi(u,v) = \begin{pmatrix} au \\ bv \end{pmatrix}$ $\Rightarrow D\Phi = \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix} \Rightarrow \det D\Phi = ab$

$$\Rightarrow \int_E dx = \int_{B_1(0)} ab\, d(u,v) = ab\pi.$$

• E.g., Gaussian integral: $I = \int_{-\infty}^{\infty} e^{-x^2} dx$

Trick: $I^2 = \int_{-\infty}^{\infty} e^{-x^2} dx \int_{-\infty}^{\infty} e^{-y^2} dy = \int_{\mathbb{R}^2} e^{-x^2-y^2} d(x,y) := \lim_{R\to\infty} \int_{B_R(0)} e^{-x^2-y^2} d(x,y)$

$$= \int_0^{\infty} \int_0^{2\pi} e^{-r^2} d\varphi \, r \, dr$$

$$= 2\pi \int_0^{\infty} \underbrace{e^{-r^2} r}_{= \frac{-1}{2}\left(\frac{d}{dr} e^{-r^2}\right)} dr = -\pi \, e^{-r^2}\Big|_0^{\infty} = \pi$$

$$\Rightarrow \int_{-\infty}^{\infty} e^{-x^2} dx = \sqrt{\pi} .$$

Other very important coordinates:

• Spherical coordinates: $\boxed{\Phi(r,\varphi,\theta) = \begin{pmatrix} r\cos\varphi \sin\theta \\ r\sin\varphi \sin\theta \\ r\cos\theta \end{pmatrix}}$, $r \geq 0$, $0 \leq \varphi < 2\pi$, $0 \leq \theta \leq \pi$

$$\Rightarrow |\det \Phi| = r^2 \sin\theta$$

• Cylindrical coordinates: $\boxed{\Phi(\rho,\varphi,z) = \begin{pmatrix} \rho\cos\varphi \\ \rho\sin\varphi \\ z \end{pmatrix}}$, $\rho \geq 0$, $0 \leq \varphi < 2\pi$, $z \in \mathbb{R}$

$$\Rightarrow |\det \Phi| = \rho$$

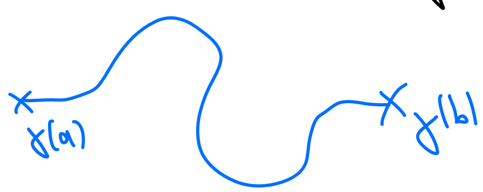
Next: We consider integrals along curves and surfaces and their relation to Riemann integrals and each other. This will lead us to generalizations of the Fundamental Theorem of Calculus. (E.g., $\int_{\text{curve } \gamma} F \, dx$ depends only on endpoints $\gamma(a)$ and $\gamma(b)$. E.g., $\underbrace{\int_D \nabla \cdot G \, dS = \int_{\partial D} G \cdot n \, ds}_{\text{integral over } D \text{ depends only on } \partial D \text{!}}$.)

Applications: Force fields, electrodynamics, ...

3.3 Line Integrals

We first introduce curves and their length.

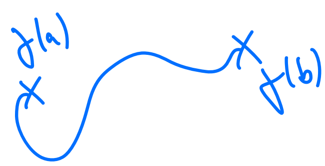
Definition:
Any continuous function $\gamma : [a,b] \to \mathbb{R}^n$ is called an **oriented curve** (or path).
i.e., $\gamma \in C([a,b], \mathbb{R}^n)$



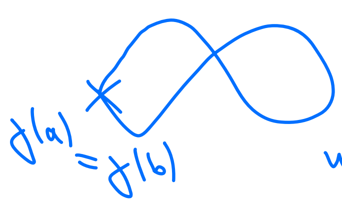
A few important types of curves:

- If $\gamma(a) = \gamma(b)$, the curve is **closed**.
- If $\gamma : [a,b] \to \mathbb{R}^n$ is injective, the curve is **simple**.

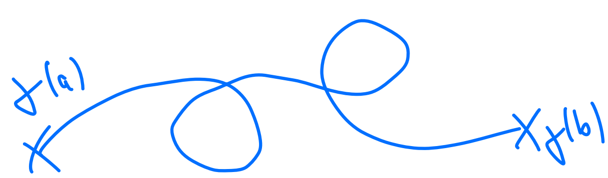
simple closed curve



simple, not closed



not simple, closed



not simple, not closed

- Two curves γ and ρ are called **equivalent** if there is a continuous, monotonic, increasing h s.t. $\underbrace{\gamma = \rho \circ h}_{\text{"going through the curve with a different velocity"}}$ (i.e., the images of γ and $\rho \circ h$ are the same).

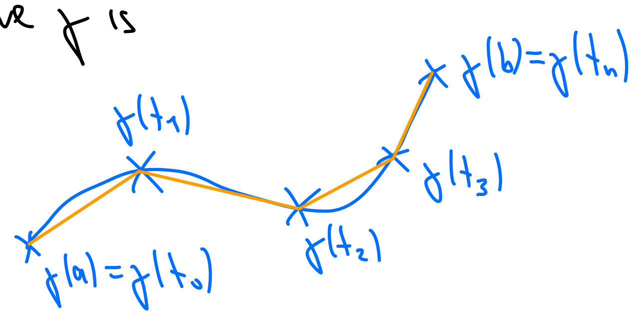
Next: length of a curve (note that it will turn out that not all curves have a length!)

Let τ be a partition of $[a,b]$ with $a = t_0 < t_1 < t_2 < \ldots < t_n := b$, and let

$$\lambda(\tau) := \max_{i=1,\ldots,n} \underbrace{|t_i - t_{i-1}|}_{=: \Delta t_i}$$

Then an approximation to the length of a curve γ is

$$\Lambda(\tau, \gamma) := \sum_{i=1}^{n} \|\gamma(t_i) - \gamma(t_{i-1})\|.$$



length of γ $\approx$ sum of lengths of line segments

Definition: The $\boxed{\text{length of the curve}}$ $\gamma \in C([a,b])$ is defined as

$$\Lambda(\gamma) = \sup_{\tau} \Lambda(\tau, \gamma).$$

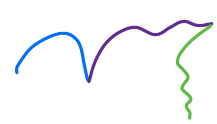
If $\Lambda(\gamma) < \infty$, we call γ $\boxed{\text{rectifiable}}$ ("γ has length").

We get a more concrete formula for continuously differentiable curves.

Theorem: Let $\gamma \in C^1([a,b])$. Then γ is rectifiable and

$$\boxed{\Lambda(\gamma) = \int_a^b \|\gamma'(t)\|\, dt.}$$

" $\int \|d\gamma\| = \int \| \frac{d\gamma}{dt} \| dt$ "

Note: The theorem is obviously extended to piece-wise C^1 curves. 

Proof:

"$\leq$": $$\Lambda(\tau,\gamma) = \sum_{i=1}^{n} \|\gamma(t_i) - \gamma(t_{i-1})\| = \sum_{i=1}^{n} \left\| \int_{t_{i-1}}^{t_i} \gamma'(t)\, dt \right\|$$

$$\leq \sum_{i=1}^{n} \int_{t_{i-1}}^{t_i} \|\gamma'(t)\| dt = \int_a^b \|\gamma'(t)\| dt,$$

so also $\Lambda(\gamma) = \sup_{\tau} \Lambda(\tau,\gamma) \leq \int_a^b \|\gamma'(t)\| dt$.

[Note: the "$\geq$" direction was not shown in class]

"$\geq$": Let $\varepsilon > 0$. We know that γ' is uniformly continuous $\Rightarrow \exists \delta > 0$ s.t. $\forall s,t \in [a,b]$ with $|s-t| < \delta$ we have $\|\gamma'(s) - \gamma'(t)\| < \varepsilon$.

Let τ be a partition with $\lambda(\tau) < \delta$.

Then $\|\gamma'(t)\| \leq \|\gamma'(t_i)\| + \varepsilon \quad \forall t \in [t_{i-1}, t_i]$

$$\Rightarrow \int_a^b \|\gamma'(t)\| dt = \sum_{i=1}^{n} \int_{t_{i-1}}^{t_i} \|\gamma'(t)\| dt \leq \sum_{i=1}^{n} \left(\|\gamma'(t_i)\| + \varepsilon \right) \Delta t_i$$

$$= \sum_{i=1}^{n} \left(\Big\| \underbrace{\int_{t_{i-1}}^{t_i} \gamma'(t_i) dt}_{= \int_{t_{i-1}}^{t_i} \gamma'(t) dt + \int_{t_{i-1}}^{t_i} \underbrace{(\gamma'(t_i) - \gamma'(t))}_{\leq \varepsilon \text{ in abs. value}} dt} \Big\| + \varepsilon \Delta t_i \right)$$

$$\leq \sum_{i=1}^{n} \left(\Big\| \underbrace{\int_{t_{i-1}}^{t_i} \gamma'(t) dt}_{= \gamma(t_i) - \gamma(t_{i-1})} \Big\| + 2\varepsilon \Delta t_i \right)$$

$$= \underbrace{\Lambda(\tau,\gamma)}_{\leq \Lambda(\gamma)} + 2\varepsilon(b-a)$$

Since ε was arbitrary (arbitrarily small), we find $\int_a^b \|\gamma'(t)\| dt \leq \Lambda(\gamma)$. $\square$

Note that for $\gamma \in C^1$, the length $\Lambda(\gamma)$ is independent of the parameterization: If $\gamma = \varrho \circ h$, $h \in C^1$ increasing, then

$$\int_a^b \|\gamma'(t)\| dt = \int_a^b \left\| \frac{d}{dt} \varrho(h(t)) \right\| dt \underset{\text{chain rule}}{=} \int_a^b \|\varrho'(h(t))\| \, h'(t) \, dt \underset{\text{substitution}}{=} \int_{h(a)}^{h(b)} \|\varrho'(u)\| du.$$

$(u = h(t) \Rightarrow du = h'(t) dt)$