
Last time: Let $F \in C(D, \mathbb{R}^n)$ ($D \subset \mathbb{R}^n$ a domain). Then:

F conservative $\iff \int_\gamma F\,dx$ depends only on $\gamma(a), \gamma(b)$ for any C^1 curve $\gamma : [a,b] \to \mathbb{R}^n$

$\iff \oint_\gamma F\,dx = 0 \quad \forall$ closed curves γ

From physics: work $\int_\gamma F\,dx$ should only depend on $\gamma(a), \gamma(b)$ if F comes from a potential, i.e., $F = \nabla \phi$, $\phi : \mathbb{R}^n \to \mathbb{R}$. Indeed:

Theorem: $F \in C(D, \mathbb{R}^n)$ ($D \subset \mathbb{R}^n$ a domain) is conservative if and only if $\exists\, \phi \in C^1(D, \mathbb{R})$ s.t. $F = \nabla \phi$. (ϕ is called a **potential** for F.)

Proof:

"$\Leftarrow$" This direction is a direct computation:

$$\int_\gamma F\,dx := \int_a^b F(\gamma(t)) \cdot \gamma'(t)\,dt = \int_a^b (\nabla\phi)(\gamma(t))\,\gamma'(t)\,dt \underset{\text{chain rule}}{=} \int_a^b \frac{d}{dt}\big(\phi(\gamma(t))\big)\,dt \underset{\text{Fundamental Theorem of Calculus}}{=} \phi(\gamma(a)) - \phi(\gamma(b)).$$

$\Rightarrow F$ conservative.

(Note: For γ piecewise smooth we split it into a sum of C^1 curves first.)

"$\Rightarrow$" We construct ϕ directly. If F conservative, we fix some $x_0 \in D$ and define

$\phi(x) = \int_\gamma F\,dx$, where γ is any C^1 curve with $\gamma(a) = x_0$, $\gamma(b) = x$. (If F were not conservative, ϕ would not just be a fct. of x.)

Then we check:

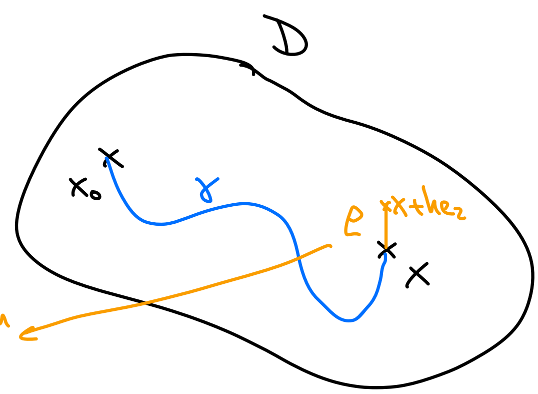
$$\phi(x + he_i) = \int_{\gamma \cup e} F\,dx = \int_\gamma F\,dx + \int_e F\,dx$$

h small enough

$$= \phi(x) + \int_0^h F(x + te_i) \cdot e_i\,dt$$

$$= \phi(x) + \int_0^h F_i(x + te_i)\,dt$$

$e(t) = x + te_i$, $0 \le t \le h$
$\Rightarrow e'(t) = e_i$



$$\Rightarrow \frac{\partial \phi}{\partial x_i} := \lim_{h\to 0} \frac{\phi(x+he_i)-\phi(x)}{h} = \frac{d}{dh}\phi(x+he_i)\Big|_{h=0} = F_i(x+he_i)\Big|_{h=0} = F_i(x),$$

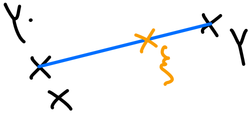
i.e., $\nabla\phi = F$.

Since F continuous, $\phi \in C^1$. $\square$

Note:

- If $\phi(x)-\psi(x) = \text{const}$ on D, then $\nabla\phi = \nabla\psi$ on D.
- Let $F = \nabla\phi = \nabla\psi$. For any $\theta \in C^1(D)$, the mean value thm. tells us that

  $\theta(x)-\theta(y) = \nabla\theta(\xi)(x-y)$ for some ξ on straight line between x and y.

  Since D is a domain, it is connected, i.e., any two points can be connected by a polygonal path.

  So for $\theta = \phi-\psi$ we have $\nabla\theta = 0$ and thus $\theta = \phi - \psi = \text{const}$ along any straight line segment, i.e., $\phi-\psi = \text{const}$ on D.

$$\Rightarrow F = \nabla\phi = \nabla\psi \text{ on } D \iff \phi-\psi = \text{const on } D.$$

An immediate consequence of the thm. above is:

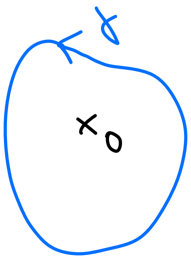
<u>Corollary:</u> If $F \in C^1(D,\mathbb{R}^n)$ ($D \subset \mathbb{R}^n$ a domain) is conservative, then the derivative DF is symmetric.

<u>Proof:</u> F conservative $\Rightarrow F = \nabla\phi \Rightarrow DF = D(\nabla\phi) = H_\phi$ i.e., $(DF)_{ij} = \frac{\partial\phi}{\partial x_i \partial x_j}$, which is symmetric (for $F \in C^1$, i.e., $\phi \in C^2$). $\square$

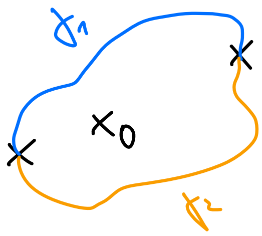
Next: Is "DF symmetric" also sufficient for F to be conservative?

Not always...

The problem can be the topological shape of the domain $D = \mathbb{R}^2 \setminus \{0\}$. A "hole" at, e.g., 0, makes it not "simply connected", where "simply connected" means: any closed curve can be continuously contracted to a point (or equivalently: any two paths with same start/end points can be continuously deformed into each other, keeping the start/end points fixed).



If 0 is missing, γ cannot be cont. deformed to a point.

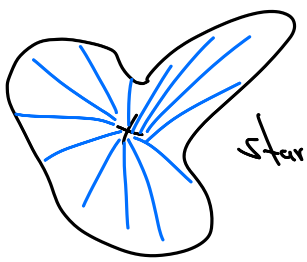


γ_1 cannot be cont. deformed into γ_2 (keeping start/end points fixed) if 0 is missing

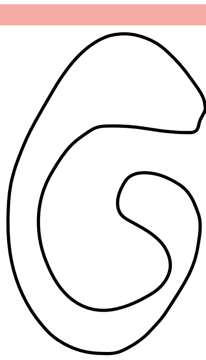
Generally, if D is simply connected, then $F: D \to \mathbb{R}^n$, $F \in C^1$ with DF symmetric implies that F is conservative.

Here, we prove this for a special case.

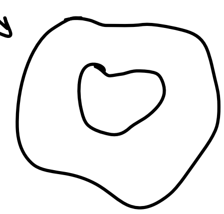
Definition: $D \subset \mathbb{R}^n$ is called star-shaped if $\exists p \in D$ such that any $x \in D$ can be connected to p by a straight line segment. (Such p are called "star center".)



star-shaped



not star-shaped



(Any non-empty convex set is star-shaped.)

Theorem: Let $D \subset \mathbb{R}^n$ be star-shaped. Then:

$$F \in C^1(D, \mathbb{R}^n) \text{ conservative} \iff DF \text{ symmetric}$$

Proof: We need to show "$\Leftarrow$". Let 0 be the center of D (without loss of generality).

Let γ be the straight line segment from 0 to x, i.e., $\gamma: [0,1] \to \mathbb{R}^n$, $t \mapsto tx$. Then we def.

$$\phi(x) := \int_\gamma F \cdot dx = \int_0^1 \underbrace{F(tx)}_{F(\gamma(t))} \cdot \underbrace{x}_{\gamma'(t)}\, dt .$$

$$\Rightarrow \frac{\partial \phi}{\partial x_i} \underset{\text{product rule}}{=} \int_0^1 \Big(\underbrace{\frac{\partial F}{\partial x_i}(tx)\, t \cdot x + F(tx) \cdot e_i}_{} \Big)\, dt = \int_0^1 \frac{d}{dt}\big(tF_i(tx)\big)\, dt = F_i(x) - 0 .$$

$$= \sum_j \underbrace{\frac{\partial F_j}{\partial x_i}}_{= \frac{\partial F_i}{\partial x_j}(tx) \text{ by assumption that } DF \text{ is symmetric}}(tx)\, x_j t + F_i(tx) = \frac{d}{dt}\big(tF_i(tx)\big)$$

$\Rightarrow D\phi = F$, i.e., F is conservative. $\square$