Recall:

$F \in C(D, \mathbb{R}^n)$ conservative $\iff \int_\gamma F\,dx$ depends only on $\gamma(a), \gamma(b)$ for any $\gamma \in C^1([a,b], D)$

$\iff \int_\gamma F\,dx = 0 \quad \forall$ closed curves γ

$\iff \exists\, \phi \in C^1(D, \mathbb{R})$ s.t. $F = \nabla \phi$

For D a star-shaped domain:

$F \in C^1(D, \mathbb{R}^n)$ conservative $\iff DF$ symmetric

Examples:

- let $F(x,y) = \left(\frac{y^2}{1+x^2}, 2y \arctan x \right)$. Task: Compute $\int_\gamma F\,dx$, e.g., for γ an ellipse.

  Here, $\frac{\partial F_1}{\partial y} = \frac{2y}{1+x^2}$, and $\frac{\partial F_2}{\partial x} = 2y \frac{1}{1+x^2}$ so DF is symmetric on $\mathbb{R}^2$, which is star-shaped.

  $\Rightarrow \int_\gamma F\,dx = 0$ for any closed γ.

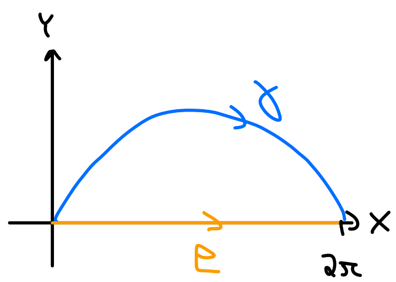
- let $\gamma : [0, 2\pi] \to \mathbb{R}^2$, $t \mapsto (t - \sin t, 1 - \cos t)$ be the cycloid.

  let $F = \frac{2}{1+x^2+y^2}(x,y)$ and compute $\int_\gamma F \cdot dx$.

  $$\int_\gamma F \cdot dx = \int_0^{2\pi} F(\gamma(t))\,\gamma'(t)\,dt = \int_0^{2\pi} \frac{2}{1+(t-\sin t)^2 + (1-\cos t)^2} \begin{pmatrix} t - \sin t \\ 1 - \cos t \end{pmatrix} \begin{pmatrix} 1 - \cos t \\ \sin t \end{pmatrix} dt = \text{very lengthy} \ldots$$

But: F is conservative on $\mathbb{R}^2$: $\frac{\partial F_1}{\partial y} = \frac{-4xy}{(1+x^2+y^2)^2} = \frac{\partial F_2}{\partial x}$

let $e : [0, 2\pi] \to \mathbb{R}^2$, $t \mapsto (t, 0)$.



$$\Rightarrow \int_\gamma \underbrace{F \cdot dx}_{\text{complicated to compute}} = \int_{e} \underbrace{F \cdot dx}_{\text{easier to compute}} = \int_0^{2\pi} F(e(t))e'(t)\,dt = \int_0^{2\pi} \frac{2}{1+t^2}(t,0)\cdot\begin{pmatrix}1\\0\end{pmatrix} dt = \int_0^{2\pi} \frac{2t}{1+t^2}\,dt$$

$$= \ln(1+t^2)\Big|_0^{2\pi} = \ln(1+4\pi^2) - \ln(1) = \ln(1+4\pi^2).$$

Alternatively: We could have guessed and then checked that $\Phi(x,y) = \ln(1+x^2+y^2)$ is a potential of $F \Rightarrow \int_\gamma F\cdot dx = \ln(1+\gamma_1(b)^2+\gamma_2(b)^2) - \ln(1+\gamma_1(a)^2+\gamma_2(a)^2)$

$$= \ln(1+4\pi^2) - \ln(1) = \ln(1+4\pi^2).$$

Example to show that "DF symmetric" is not sufficient for F to be conservative:

$F = \left(-\frac{y}{x^2+y^2}, \frac{x}{x^2+y^2}\right)$ on $D = \mathbb{R}^2 \setminus \{0\} \Rightarrow D$ not star-shaped

Here: $\dfrac{\partial F_1}{\partial y} = \dfrac{-1\cdot(x^2+y^2)+y2y}{(x^2+y^2)^2} = \dfrac{y^2-x^2}{(x^2+y^2)^2}$,

and $\dfrac{\partial F_2}{\partial x} = \dfrac{1\cdot(x^2+y^2)-x2x}{(x^2+y^2)^2} = \dfrac{y^2-x^2}{(x^2+y^2)^2}$, so DF is symmetric.

But: let $\gamma : [0,2\pi] \to \mathbb{R}^2$, $t \mapsto (\cos t, \sin t)$ (unit circle).

Then $\displaystyle\int_\gamma F\cdot dx = \int_0^{2\pi} \underbrace{(-\sin t, \cos t)}_{F(\gamma(t))} \underbrace{\begin{pmatrix}-\sin t\\ \cos t\end{pmatrix}}_{\gamma'(t)} dt = \int_0^{2\pi} \underbrace{(\sin^2 t+\cos^2 t)}_{=1}\,dt = 2\pi.$

$\Rightarrow F$ not conservative!

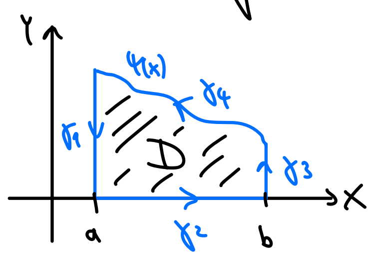
3.4 Green's Theorem

Green's thm. relates integrals over bounded closed domains $\overline{D} \subset \mathbb{R}^2$ to line integrals over the boundary ∂D.

The formula and proof are based on the following computation:

Consider

- an x-normal domain $D = \{(x,y) : x \in (a,b),\ 0 \le y \le \psi(x)\}$,
- a C^1 vector field $F = (f,g)$,
- a curve γ:



i.e., $\gamma = \partial D$ (going anti-clockwise).

Then $\displaystyle\int_D \left(\frac{\partial g}{\partial x} - \frac{\partial f}{\partial y}\right) dS$

$$= \int_a^b \int_0^{\psi(x)} \left(\frac{\partial g}{\partial x}\right) dy\, dx - \int_a^b \overbrace{\int_0^{\psi(x)} \left(\frac{\partial f}{\partial y}\right) dy}^{= f(x,0) - f(x,\psi(x))} dx$$

$$= \underbrace{\int_a^b \frac{\partial}{\partial x}\left[\int_0^{\psi(x)} g\, dy\right] dx}_{= \int_0^{\psi(b)} g(b,y)\,dy - \int_0^{\psi(a)} g(a,y)\,dy} - \int_a^b g(x,\psi(x))\,\psi'(x)\,dx + \int_a^b f(x,0)\,dx - \int_a^b f(x,\psi(x))$$

On the other hand: $\displaystyle\int F \cdot d\vec{x} = \int_{\gamma_1} \begin{pmatrix} f \\ g \end{pmatrix} \cdot d\vec{x} + \int_{\gamma_2} \begin{pmatrix} f \\ g \end{pmatrix} \cdot d\vec{x} + \int_{\gamma_3} \begin{pmatrix} f \\ g \end{pmatrix} \cdot d\vec{x} + \int_{\gamma_4} \begin{pmatrix} f \\ g \end{pmatrix} \cdot d\vec{x}$

with: $\gamma_1 : [\psi(a), 0] \to \mathbb{R}^2,\ \gamma_1(t) = (a,t)$, $\gamma_2 : [a,b] \to \mathbb{R}^2,\ \gamma_2(t) = (t,0)$,

$\gamma_3 : [0, \psi(b)] \to \mathbb{R}^2,\ \gamma_3(t) = (b,t)$, $\gamma_4 : [b,a] \to \mathbb{R}^2,\ \gamma_4(t) = (t, \psi(t))$.

$$\Rightarrow \int_{\gamma_1} \binom{f}{g} \cdot d\vec{x} = \int_{\varphi(a)}^{0} \binom{f(a,t)}{g(a,t)} \cdot \binom{0}{1} dt = -\int_{0}^{\varphi(a)} g(a,t)\,dt$$

$$\int_{\gamma_2} \binom{f}{g} \cdot d\vec{x} = \int_{a}^{b} \binom{f(t,0)}{g(t,0)} \cdot \binom{1}{0} dt = \int_{a}^{b} f(t,0)\,dt$$

$$\int_{\gamma_3} \binom{f}{g} \cdot d\vec{x} = \int_{0}^{\varphi(b)} \binom{f(b,t)}{g(b,t)} \cdot \binom{0}{1} dt = \int_{0}^{\varphi(b)} g(b,t)\,dt$$

$$\int_{\gamma_4} \binom{f}{g} \cdot d\vec{x} = \int_{b}^{a} \binom{f(t,\varphi(t))}{g(t,\varphi(t))} \cdot \binom{1}{\varphi'(t)} dt = -\int_{a}^{b} f(t,\varphi(t))\,dt - \int_{a}^{b} g(t,\varphi(t))\,\varphi'(t)\,dt$$

Thus: $\int_{D} \left(\frac{\partial g}{\partial x} - \frac{\partial f}{\partial y}\right) dS = \int_{\gamma} F \cdot d\vec{x}$

Note: Interchanging x and y gives same result for y-normal domains.

More generally, let us define:

<u>Definition:</u> $D \subset \mathbb{R}^2$ bounded is called a $\boxed{\text{regular domain}}$ if it can be decomposed into finitely many $\underbrace{\text{bi-normal}}_{\text{either x- or y-normal}}$ subdomains.

Then the result from above still holds:

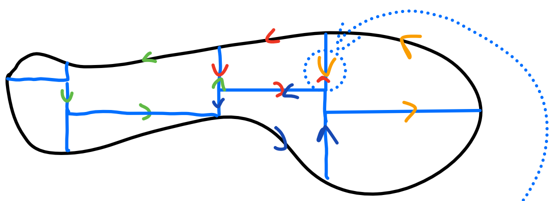
<u>Theorem (Green's theorem):</u>

Let $D \subset \mathbb{R}^2$ be a bounded regular domain, and let $F \in C^1(\bar{D}, \mathbb{R}^2)$. Then

$$\boxed{\int_{D} \left(\frac{\partial F_2}{\partial x_1} - \frac{\partial F_1}{\partial x_2}\right) dS = \int_{\partial D} F \cdot dx \qquad \text{(Green's formula)}},$$

where the line integral has anti-clockwise orientation.

Sketch of proof:



- area integrals over subdomains sum up
- interior pieces of line integrals cancel → only boundary pieces remain
- summing up yields the thm.

$\square$