Advanced Calculus and Methods of Mathematical Physics — Session 22

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Last time we derived Green's theorem:

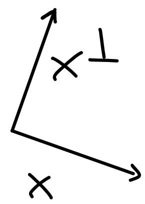
$$\int_D \left(\frac{\partial F_2}{\partial x_1} - \frac{\partial F_1}{\partial x_2} \right) dS = \underbrace{\int_{\partial D}}_{\text{with anti-clockwise orientation}} F \cdot dx \,,$$

for: • $D \subset \mathbb{R}^2$ a bounded regular domain, i.e., it can be decomposed into finitely many x-normal or y-normal subdomains,

• $F \in C^1(\overline{D}, \mathbb{R}^2)$.

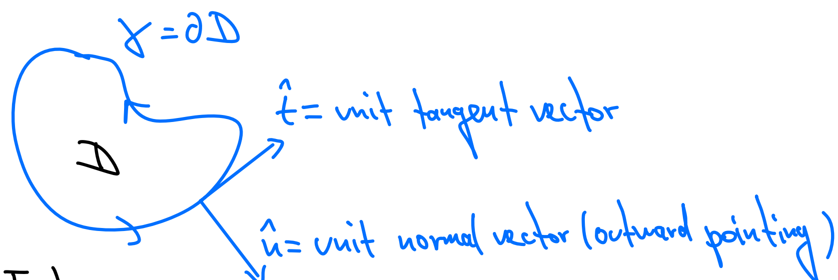
Remarks:

• For $x = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$, def. $x^\perp = \begin{pmatrix} -x_2 \\ x_1 \end{pmatrix}$.



With $\nabla^\perp := \begin{pmatrix} -\frac{\partial}{\partial x_2} \\ \frac{\partial}{\partial x_1} \end{pmatrix}$, Green's formula becomes $\boxed{\int_D \underbrace{\nabla^\perp \cdot F}_{= \mathrm{curl}_{2D} F} \, dS = \int_{\partial D} F \cdot dx.}$

• Def. vectors $\hat{n}$ and $\hat{t}$ as in the picture:



Let us define G s.t. $F = G^\perp$, i.e., $G = \begin{pmatrix} F_2 \\ -F_1 \end{pmatrix}$

Then $\int_D \nabla \cdot G \, dS = \int_D \nabla^\perp \cdot G^\perp \, dS = \int_D \nabla^\perp \cdot F \overset{\text{Green's thm.}}{=} \int_{\partial D} F \cdot dx = \int_{\partial D} \underbrace{G^\perp \cdot \hat{t}}_{= (G^\perp)^\perp \cdot \hat{t}^\perp = G \cdot \hat{n}} ds$

$$(G^\perp)^\perp = -G, \quad \hat{t}^\perp = -\hat{n}$$

$\Rightarrow \boxed{\int_D \underbrace{\nabla \cdot G}_{= \operatorname{div} G \text{ (divergence of } G)} \, dS = \int_{\partial D} G \cdot \hat{n} \, ds}$ (Divergence Theorem)

(Generalization of the Fundamental Thm. of Calculus to 2-dim. domains)

Examples:

- $F = \frac{1}{2}\begin{pmatrix}-y\\ x\end{pmatrix} = \frac{1}{2}x^{\perp} \Rightarrow \nabla^{\perp}\cdot F = \begin{pmatrix}-\frac{\partial}{\partial y}\\ \frac{\partial}{\partial x}\end{pmatrix}\frac{1}{2}\begin{pmatrix}-y\\ x\end{pmatrix} = \frac{1}{2}(1+1) = 1$

$$\Rightarrow \int_D \nabla^{\perp}\cdot F\, dS = \int_D dS = \underbrace{S(D)}_{\text{surface area of } D} \overset{\text{Green's thm.}}{=} \int_{\partial D} F\cdot dx = -\frac{1}{2}\int_{\partial D} y\,dx + \frac{1}{2}\int_{\partial D} x\,dy$$

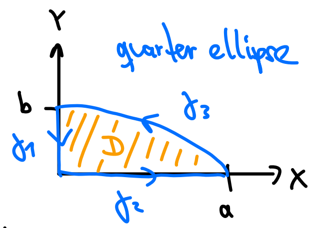
this notation is sometimes used

E.g., area of ellipse D, with ∂D parametrized by $\gamma(t) = (a\cos t, b\sin t)$, $t\in[0,2\pi]$:

$$S(D) = \int_{\partial D} F\cdot dx = \int_0^{2\pi} F(\gamma(t))\cdot\gamma'(t)\,dt = \frac{1}{2}\int_0^{2\pi} \underbrace{\gamma^{\perp}(t)\cdot\gamma'(t)}\,dt = \frac{1}{2}ab\,2\pi = \pi ab$$

$= (-b\sin t, a\cos t)\cdot(-a\sin t, b\cos t) = ab\sin^2 t + ab\cos^2 t = ab$

- $J = \int_D xy\, dS$ with $D = \{(x,y)\in\mathbb{R}^2 : \frac{x^2}{a^2} + \frac{y^2}{b^2} = 1,\ x\geq 0, y\geq 0\}$



To use Green's thm. we can choose, e.g., $F = (0, \frac{1}{2}x^2 y)$, s.t. $\nabla^{\perp}\cdot F = \begin{pmatrix}-\frac{\partial}{\partial y}\\ \frac{\partial}{\partial x}\end{pmatrix}\begin{pmatrix}0\\ \frac{1}{2}x^2 y\end{pmatrix} = xy$.

Then $J = \int_D \nabla^{\perp}\cdot F\, dS = \int_{\partial D} F\cdot dx = \underbrace{\int_{\gamma_1} F\cdot dx}_{=0\ (\text{since } x=0)} + \underbrace{\int_{\gamma_2} F\cdot dx}_{=0\ (\text{since } y=0)} + \int_{\gamma_3} F\cdot dx$

$$= \int_0^{\frac{\pi}{2}} \left(0, \frac{1}{2}(a\cos t)^2\, b\sin t\right)\cdot(-a\sin t, b\cos t)\,dt$$

$$= \int_0^{\frac{\pi}{2}} \frac{1}{2}a^2 b^2 \cos^3 t \sin t\,dt$$

$$= \frac{1}{2}a^2 b^2 \left(-\frac{1}{4}\right)\cos^4 t\,\Big|_0^{\frac{\pi}{2}}$$

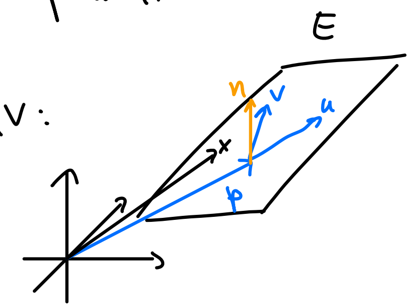
$$= \frac{a^2 b^2}{8}$$

[Note: direct computation (without using Green's thm.) is also possible here; check that it gives the same result]

3.5 Surface Integrals

First, a short review of planes, normal vectors, and the cross product.

A plane E can be parametrized by specifying vectors p, u, v:



$$E = \{x \in \mathbb{R}^3 : x = p + su + tv,\ s,t \in \mathbb{R}\}$$

Alternatively, we can write the equation of a plane as $\boxed{(p-x)\cdot n = 0, \text{ with } n := u \times v}$ the normal vector.

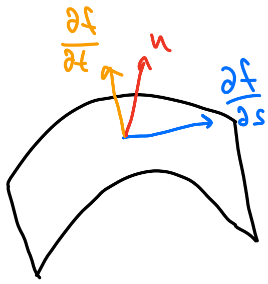
Note/recall the following properties of the cross product:

- $u \times v = -v \times u$
- $\det\begin{pmatrix} | & | & | \\ a & b & c \\ | & | & | \end{pmatrix} = a \cdot (b \times c)$
- $u \times v$ is perpendicular to u and v
- $u \times v = 0$ if u and v are linearly dependent
- The area of a parallelogram spanned by u and v is $\|u \times v\| = \|u\|\,\|v\| \sin\theta$, with θ = angle between u, v
- $\|u \times v\|^2 = \|u\|^2 \|v\|^2 - (u \cdot v)^2$

More generally, a surface $M \subset \mathbb{R}^3$ can be parametrized by a fct. $f(s,t)$, with $f \in C(\bar{U}, \mathbb{R}^3)$, $U \subset \mathbb{R}^2$ a domain. Then $M = \underbrace{\text{range of } f}_{:= \{x \in \mathbb{R}^3 :\ f(s,t) = x \text{ for some } (s,t) \in \bar{U}\}}$.

(compare to E above)

The normal vector is then a function of s and t (normal vector field).



$\rightarrow$ $\frac{\partial f}{\partial s}, \frac{\partial f}{\partial t}$ are tangent vectors (spanning the tangent plane)

$\rightarrow$ $n = \frac{\partial f}{\partial s} \times \frac{\partial f}{\partial t}$ is a normal vector (perpendicular to tangent plane)

We call M smooth if $f \in C^1(U, \mathbb{R}^3)$ and $n := \frac{\partial f}{\partial s} \times \frac{\partial f}{\partial t} \neq 0$ on U.

We define the unit normal vector $\hat{n} = \frac{n}{\|n\|}$.

Analogous to line integrals (replace γ with f, and γ' with n), we define:

- the surface area $\sigma(M) := \int_U \|n\| \, dS$, (analogous to $\Lambda(\gamma) = \int_a^b \|\gamma'(t)\| \, dt$)

  (heuristically: "Integrating up infinitesimal surface elements / little parallelograms")

- for $\Phi \in C(M, \mathbb{R})$ the surface integral $\int_M \Phi \, d\sigma := \int_U \Phi \circ f \, \|n\| \, dS$, (analogous to $\int_\gamma \Phi \, ds = \int_a^b \Phi(\gamma(t)) \|\gamma'(t)\| \, dt$)

- for $F \in C(M, \mathbb{R}^3)$ the flux integral $\int_M F \cdot \underbrace{\hat{n}}_{\text{normalized normal vector}} d\sigma := \int_U (F \circ f) \cdot n \, dS$. (analogous to $\int_\gamma F \cdot dx = \int_a^b (F \circ \gamma)(t) \cdot \gamma'(t) \, dt$)