Recall: we define a surface M by a fct. $f \in C^1(U, \mathbb{R}^3)$ ($U \subset \mathbb{R}^2$ a domain) with normal vector $n := \frac{\partial f}{\partial s} \times \frac{\partial f}{\partial t} \neq 0$. Then we define:

- the surface area $\sigma(M) := \int_U \|n\| \, dS$,
- for $\Phi \in C(M, \mathbb{R})$, the surface integral $\int_M \Phi \, d\sigma := \int_U \Phi \circ f \, \|n\| \, dS$,
- for $F \in C(M, \mathbb{R}^3)$, the flux integral $\int_M F \cdot \hat{n} \, d\sigma := \int_U (F \circ f) \cdot n \, dS$.

$\hat{n} := \frac{n}{\|n\|}$

Examples:

- Surface area of a sphere: We can choose $f(\theta, \varphi) = \begin{pmatrix} \sin\theta \cos\varphi \\ \sin\theta \sin\varphi \\ \cos\theta \end{pmatrix}$, $\varphi \in [0, 2\pi]$, $\theta \in [0, \pi]$.

(as for spherical coordinates, $r = 1$)

Then: $\frac{\partial f}{\partial \theta} = \begin{pmatrix} \cos\theta \cos\varphi \\ \cos\theta \sin\varphi \\ -\sin\theta \end{pmatrix}$, $\frac{\partial f}{\partial \varphi} = \begin{pmatrix} -\sin\theta \sin\varphi \\ \sin\theta \cos\varphi \\ 0 \end{pmatrix}$

$$\Rightarrow n = \frac{\partial f}{\partial \theta} \times \frac{\partial f}{\partial \varphi} = \begin{pmatrix} 0 + \sin^2\theta \cos\varphi \\ \sin^2\theta \sin\varphi - 0 \\ \cos\theta \sin\theta (\cos^2\varphi + \sin^2\varphi) \end{pmatrix} = \begin{pmatrix} \sin^2\theta \cos\varphi \\ \sin^2\theta \sin\varphi \\ \cos\theta \sin\theta \end{pmatrix}$$

$$\Rightarrow \|n\|^2 = \underbrace{\sin^4\theta \cos^2\varphi + \sin^4\theta \sin^2\varphi}_{= \sin^4\theta} + \cos^2\theta \sin^2\theta = \sin^2\theta (\sin^2\theta + \cos^2\theta) = \sin^2\theta$$

$$\Rightarrow \|n\| = \sin\theta$$

$$\Rightarrow \sigma(\text{sphere}) = \int_0^{2\pi} \int_0^{\pi} \sin\theta \, d\theta \, d\varphi = 2\pi \left(-\cos\theta\right)\Big|_0^{\pi} = 4\pi.$$

- $M =$ upper hemisphere of radius 1 centered at 0, $\phi(x,y,z) = (x^2+y^2)z$.

We use the same f as above with $\varphi \in [0, 2\pi]$ but $\theta \in [0, \frac{\pi}{2}]$ only.

$$
\begin{aligned}
\Rightarrow \int_M \phi \, d\sigma &= \int_0^{2\pi} \int_0^{\frac{\pi}{2}} \phi(f(\theta,\varphi)) \, \|n(\theta,\varphi)\| \, d\theta \, d\varphi \\
&= \int_0^{2\pi} \int_0^{\frac{\pi}{2}} \sin^2\theta \, (\cos^2\varphi + \sin^2\varphi) \cos\theta \, \overbrace{\sin\theta}^{=\|u\|} \, d\theta \, d\varphi \\
&= \int_0^{2\pi} \int_0^{\frac{\pi}{2}} \sin^3\theta \cos\theta \, d\theta \, d\varphi \\
&= 2\pi \left. \tfrac{1}{4} \sin^4\theta \right|_0^{\frac{\pi}{2}} \\
&= \frac{\pi}{2}.
\end{aligned}
$$

- Same M, $F = \frac{1}{x^2+y^2+z^2} \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \Rightarrow F(f(\theta,\varphi)) = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$

, since $(\sin\theta\cos\varphi)^2 + (\sin\theta\sin\varphi)^2 + (\cos\theta)^2 = 1$.

$$
\Rightarrow \int_M F \cdot \hat{n} \, d\sigma = \int_0^{2\pi} \int_0^{\frac{\pi}{2}} \underbrace{F(f(\theta,\varphi))}_{= \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}} \cdot \underbrace{n(\theta,\varphi)}_{= \begin{pmatrix} \sin^2\theta \cos\varphi \\ \sin^2\theta \sin\varphi \\ \cos\theta \sin\theta \end{pmatrix}} \, d\theta \, d\varphi
$$

outward pointing

$$
\begin{aligned}
&= \int_0^{2\pi} \int_0^{\frac{\pi}{2}} \left(\sin^2\theta \cos\varphi + \sin^2\theta \sin\varphi + \cos\theta \sin\theta \right) d\theta \, d\varphi \\
&= \int_0^{\frac{\pi}{2}} \sin^2\theta \, d\theta \underbrace{\int_0^{2\pi} (\cos\varphi + \sin\varphi) \, d\varphi}_{=0} + 2\pi \left. \tfrac{1}{2} \sin^2\theta \right|_0^{\frac{\pi}{2}} \\
&= \pi
\end{aligned}
$$

Next, we relate flux integrals to volume integrals ("Green's theorem one dimension higher")

3.6 Divergence Theorem

Let us state and discuss the result; we omit proofs.

The divergence thm. holds in any dimension:

Theorem (Divergence Theorem; also called Gauß' Theorem):

Let $D \subset \mathbb{R}^n$ be a domain and let $V \subset D$ s.t.

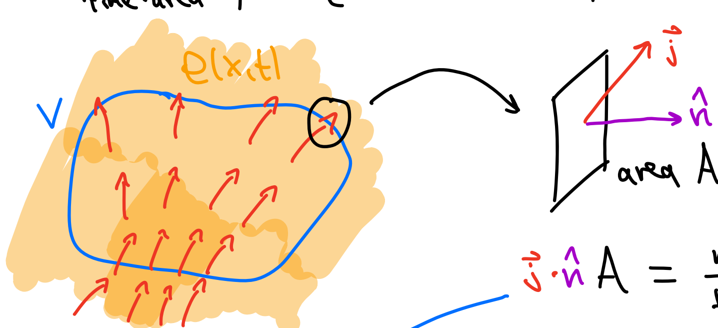
- $\bar{V} \subset D$, and $\bar{V}$ bounded and regular,
- ∂V has non-vanishing piece-wise continuous normal field n,
- $F \in C^1(D, \mathbb{R}^n)$.

Then $\boxed{\int_{\partial V} F \cdot \hat{n} \, d\sigma = \int_V \underbrace{\nabla \cdot F}_{= \operatorname{div} F \text{ ("divergence of } F\text{")}} \, dx}$, with $\hat{n}$ the outward pointing unit normal vector.

Some interpretation / connection to physics:

Let

- $\rho(x,t)$ = density at point x at time t (e.g., mass density or probability density).
- $m_V(t) = \int_V \rho(x,t) \, d^3x$ = total mass in domain $V \subset \mathbb{R}^3$.
- $\vec{j}(x,t)$ = flux density (units: $\frac{\text{mass}}{\text{time} \cdot \text{area}}$) $= \rho(x,t) \cdot \vec{u}(x,t)$, with $\vec{u}(x,t)$ the velocity vector field.



$\vec{j} \cdot \hat{n} \, A = \frac{\text{mass}}{\text{time}}$ passing through surface integrate up small area elements $\vec{j}\cdot\hat{n}\,A$; minus sign since mass flows out

Change of mass in time is $\displaystyle \frac{dm_V(t)}{dt} = -\int_{\partial V} \vec{j}\cdot\hat{n}\, d\sigma \overset{\text{Gauß}}{=} -\int_V \operatorname{div}\vec{j}\, d^3x$

by def. of $m_V(t)$

$$= \int_V \frac{\partial\rho(x,t)}{\partial t}\, d^3x$$

$\Rightarrow$ Since this holds $\forall$ domains V, we have deduced the

$$\boxed{\text{continuity equation } \frac{\partial\rho}{\partial t} + \operatorname{div}\vec{j} = 0.}$$

For heat ("Fourier's law") or other diffusion processes ("Fick's law") we have

$\vec{j} = -\vec{\nabla}\rho$.

$$\Rightarrow \operatorname{div}\vec{j} = -\vec{\nabla}\cdot\vec{\nabla}\rho = -\underbrace{\Delta}\rho,$$

$\vec{\nabla}\cdot\vec{\nabla} = \frac{\partial^2}{\partial x_1^2} + \dots \frac{\partial^2}{\partial x_n^2}$ = Laplace operator

which leads to the $\boxed{\text{heat eq. or diffusion eq. } \frac{\partial\rho}{\partial t} = \Delta\rho.}$

Another result is a generalization of Green's thm. to general surfaces M.

Theorem (Stokes' theorem):

Let $D \subset \mathbb{R}^3$ be a domain, let $M \subset D$ a smooth surface that is bounded and orientable, and let ∂M have smooth parametrization with orientation anti-clockwise w.r.t. to the normal field of M. Let $F \in C^1(D, \mathbb{R}^3)$.

Then

$$\boxed{\int_{\partial M} F\cdot dx = \int_M \underbrace{(\nabla\times F)}_{=:\operatorname{curl} F}\cdot\hat{n}\, d\sigma.}$$

Note:

- "Orientable" means that a normal vector field can be chosen consistently (always the case for $M := \text{range } f$).
- For flat surfaces we recover Green's thm.: set $F = \begin{pmatrix} F_1 \\ F_2 \\ 0 \end{pmatrix}$, $\hat{n} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$.

Before we discuss examples, a few interesting implications.

<u>Corollary:</u> With the same notation as in Stokes' thm., assume that M has no boundary. Then $\int_M (\nabla \times F) \cdot \hat{n} \, d\sigma = 0$ for any $F \in C^1(D, \mathbb{R}^3)$.

<u>Corollary:</u> Let $F \in C^1(D, \mathbb{R}^3)$, $D \subset \mathbb{R}^3$ a simply connected domain. Then

$$F \text{ conservative} \iff \nabla \times F = 0.$$

<u>Sketch of proof:</u> "$\Rightarrow$" If $F = \nabla \phi$, then $\nabla \times F = \nabla \times \nabla \phi = 0$, see homework.

"$\Leftarrow$" Let γ be a closed curve. Under the stated assumptions one can show that there is a "capping surface" M s.t. $\partial M = \gamma$.

Then $\int_\gamma F \cdot dx = \int_M (\nabla \times F) \cdot \hat{n} \, d\sigma = 0 \Rightarrow F$ conservative. $\square$