Last time:

- Divergence/Gauß theorem: (V a bounded regular domain)

$$\int_V \nabla\cdot F\, dx = \int_{\partial V} F\cdot\hat{n}\, d\sigma \quad , \text{ with } \hat{n} \text{ the outward pointing unit normal vector.}$$

- Stokes' theorem: (M a smooth orientable surface)

$$\int_M (\nabla\times F)\cdot\hat{n}\, d\sigma = \int_{\partial M} F\cdot dx .$$

Corollary: Let $F\in C^1(D,\mathbb{R}^3)$, $D\subset\mathbb{R}^3$ a simply connected domain. Then

$$F \text{ conservative} \iff \nabla\times F = 0.$$

Sketch of proof: "$\Rightarrow$" If $F=\nabla\phi$, then $\nabla\times F = \nabla\times\nabla\phi = 0$ (partial derivatives commute)

"$\Leftarrow$" Let γ be a closed curve. Under the stated assumptions one can show that there is a "capping surface" M s.t. $\partial M=\gamma$.
Then $\int_\gamma F\cdot dx = \int_M (\nabla\times F)\cdot\hat{n}\, d\sigma = 0 \Rightarrow F$ conservative. $\square$

Examples:

- Let $F(x,y,z)=(x^3,y^3,z^3)$. We compute the flux $\Phi_M(F)$ through the upper hemisphere $M:=\{(x,y,z)\in\mathbb{R}^3 : x^2+y^2+z^2=R^2, z\geq 0\}$ with radius $R>0$.

$M':=\{(x,y,0)\in\mathbb{R}^3 : x^2+y^2\leq R^2\}$ the "bottom"

M and M' enclose the volume $V = \{(x,y,z) \in \mathbb{R}^3 : x^2+y^2+z^2 \leq R^2, z \geq 0\}$.

Then
$$\Phi_M(F) := \int_M F\cdot\hat{n}\, d\sigma = \int_{M\cup M'} F\cdot\hat{n}\, d\sigma - \int_{M'} \underbrace{F\cdot\hat{n}}_{=F\cdot\begin{pmatrix}0\\0\\-1\end{pmatrix}\big|_{z=0} = -F_3|_{z=0} = 0} d\sigma$$

Gauss
$$= \int_V \operatorname{div} F\, d^3x$$
$$= 3\int_V (x^2+y^2+z^2)\, d^3x$$

spherical coordinates
$$= 3\int_0^{2\pi}\int_0^{\frac{\pi}{2}}\int_0^R r^2\, r^2 \sin\theta\, dr\, d\theta\, d\varphi$$
$$= 3\cdot 2\pi \cdot 1 \cdot \frac{R^5}{5}$$
$$= \frac{6\pi}{5} R^5.$$

- Example for Stokes' thm.: see homework

Another application: Maxwell's equations in differential and integral form

- $V \subset \mathbb{R}^3$ a bounded volume with closed boundary ∂V
- $M \subset \mathbb{R}^3$ a surface with closed boundary curve ∂M
- ρ = charge density, $Q = \int_V \rho\, d^3x$ the total electric charge within V
- J = el. current density, $I = \int_M J\cdot\hat{n}\, d\sigma$ the el. current passing through M
- E = electric field, B = magnetic field
- c = speed of light

Differential version | Integral version

$$\nabla\cdot E = 4\pi\rho \quad \overset{\text{Gauß}}{\longleftrightarrow} \quad \int_{\partial V} E\cdot\hat{n}\,d\sigma = 4\pi Q$$

$$\nabla\cdot B = 0 \quad \overset{\text{Gauß}}{\longleftrightarrow} \quad \int_{\partial V} B\cdot\hat{n}\,d\sigma = 0$$

$$\nabla\times E = -\frac{1}{c}\frac{\partial B}{\partial t} \quad \overset{\text{Stokes}}{\longleftrightarrow} \quad \int_{\partial M} E\cdot dx = -\frac{1}{c}\frac{d}{dt}\int_{M} B\cdot\hat{n}\,d\sigma$$

$$\nabla\times B = \frac{1}{c}\left(4\pi J + \frac{\partial E}{\partial t}\right) \quad \overset{\text{Stokes}}{\longleftrightarrow} \quad \int_{\partial M} B\cdot dx = \frac{1}{c}\left(4\pi I + \frac{d}{dt}\int_{M} E\cdot d\sigma\right)$$

Note: $\frac{\partial^2 E}{\partial t^2} = c\frac{\partial}{\partial t}\nabla\times B - 4\pi\frac{\partial J}{\partial t}$

$J=0$ in vacuum

$$= c\,\nabla\times\frac{\partial B}{\partial t}$$

$$= -c^2\nabla\times(\nabla\times E)$$

$$= -c^2\nabla\underbrace{(\nabla\cdot E)}_{=4\pi\rho=0 \text{ in vacuum}} + c^2\nabla^2 E$$

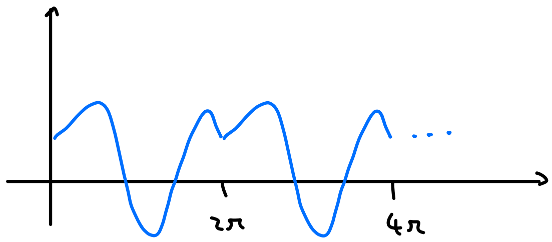
$\Longrightarrow$ In vacuum we get the wave eq. $\frac{1}{c^2}\frac{\partial^2 E}{\partial t^2} - \Delta E = 0$

($\Delta := \nabla^2 =$ Laplace operator)

4. Fourier Series

We consider 2π-periodic functions, i.e., $f(x+2\pi)=f(x)$

(L-periodic for any $0 \neq L \in \mathbb{R}$ works analogously).



Fourier series:
- idea: decompose functions into "pure frequencies" (e.g., signals)
- works also for non-differentiable functions (as opposed to Taylor series)

Let us just consider one period, i.e., $f: [0,2\pi] \longrightarrow \mathbb{C}$, $f(0)=f(2\pi)$.

We assume f is Riemann-integrable on $[0,2\pi]$.

Then the $\boxed{\text{Fourier series of } f \text{ is defined as } F_f(x) := \sum_{k=-\infty}^{\infty} \underbrace{\hat{f}_k}_{=\text{Fourier coefficients}} \overbrace{e^{ikx}}^{=\cos kx + i \sin kx}.}$

Note: $e_k(x) := e^{ikx}$ plays the role of a basis function.

Let us introduce the inner product $\langle f,g \rangle := \frac{1}{2\pi} \int_0^{2\pi} \overline{f(x)}\, g(x)\,dx$ and norm $\|f\| = \sqrt{\langle f,f \rangle}$.

$$\text{Then } \langle e_j, e_k \rangle = \frac{1}{2\pi}\int_0^{2\pi} \overline{e_j(x)}\, e_k(x)\,dx = \frac{1}{2\pi}\int_0^{2\pi} e^{-ijx} e^{ikx}\,dx = \frac{1}{2\pi}\begin{cases} \frac{1}{i(k-j)} e^{i(k-j)x}\Big|_0^{2\pi} = 0 & k \neq j \\ 2\pi & k = j \end{cases}$$

$$= \underbrace{\delta_{jk}}_{\text{Kronecker delta}} = \begin{cases} 1 & \text{if } j=k \\ 0 & \text{if } j \neq k \end{cases}$$

Now assuming $F_f(x)$ converges uniformly to $f(x)$, we have

$$\langle e_j, f\rangle = \langle e_j, \sum_{k=-\infty}^{\infty} \hat{f}_k e_k\rangle \underset{\substack{\uparrow \\ \text{uniform} \\ \text{convergence}}}{=} \sum_{k=-\infty}^{\infty} \hat{f}_k \underbrace{\langle e_j, e_k\rangle}_{\delta_{jk}} = \hat{f}_j .$$

So far we know: If $f(x) := \sum_{k=-\infty}^{\infty} \hat{f}_k e^{ikx}$ is uniformly convergent, then $\hat{f}_k = \frac{1}{2\pi}\int_0^{2\pi} e^{-ikx} f(x)dx$.

But we can define $\hat{f}_k$ for any Riemann integrable f.

So generally, we define the $\boxed{\text{Fourier transform of } f \text{ as } \hat{f}_k := \langle e_k, f\rangle = \frac{1}{2\pi}\int_0^{2\pi} e^{-ikx} f(x)dx.}$

Question for next time:

Does $\sum_{k=-\infty}^{\infty} \hat{f}_k e^{ikx}$ always converge to $f(x)$, and if yes, in what sense?