Advanced Calculus and Methods of Mathematical Physics — Session 25

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Consider $f: [0, 2\pi] \longrightarrow \mathbb{C}$, $f(0) = f(2\pi)$, f Riemann integrable.

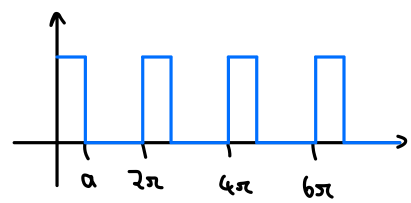
We define the Fourier transform of f as $\hat{f}_k := \frac{1}{2\pi} \int_0^{2\pi} e^{-ikx} f(x)\,dx$.

The Fourier series of f is def. as $F_f(x) := \sum_{k=-\infty}^{\infty} \hat{f}_k e^{ikx}$.

Last time: If F_f is uniformly convergent, then $F_f(x) = f(x)$.

But often $F_f(x)$ and $f(x)$ don't agree everywhere.

Example A: $f(x) = \begin{cases} 1 & \text{for } x \in [0, a) \\ 0 & \text{for } x \in [a, 2\pi) \end{cases}$



We find $\cdot\ \hat{f}_0 = \frac{1}{2\pi} \int_0^{2\pi} f(x)\,dx = \frac{a}{2\pi}$

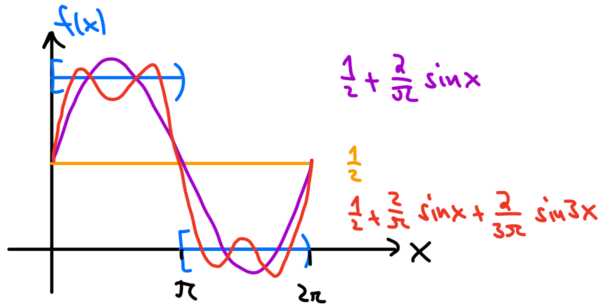
$\cdot$ For $k \neq 0$: $\hat{f}_k = \frac{1}{2\pi} \int_0^{a} e^{-ikx} f(x)\,dx = \frac{1}{2\pi} \int_0^{a} e^{-ikx}\,dx = \frac{1}{2\pi} \frac{1}{(-ik)} e^{-ikx} \Big|_0^a$

$$= \frac{i}{2\pi k} \left(e^{-ika} - 1 \right)$$

E.g., for $a = \pi$, we have $\hat{f}_k = \frac{i}{2\pi k} \left(e^{-i\pi k} - 1 \right) = \frac{i}{2\pi k} \left((-1)^k - 1 \right) = \begin{cases} 0 & \text{for } k \text{ even} \\ \frac{-i}{\pi k} & \text{for } k \text{ odd} \end{cases}$

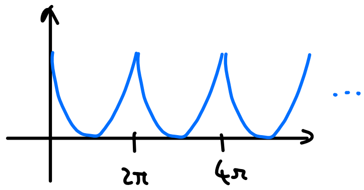
and $$F_f(x) = \sum_{k=-\infty}^{\infty} \hat{f}_k e^{ikx} = \underbrace{\frac{1}{2}}_{\hat{f}_0} + \sum_{\substack{k=1 \\ k \text{ odd}}}^{\infty} \frac{(-i)}{\pi k} e^{ikx} + \underbrace{\sum_{\substack{k=-\infty \\ k \text{ odd}}}^{-1} \frac{(-i)}{\pi k} e^{ikx}}_{= -\sum_{\substack{k=1 \\ k \text{ odd}}}^{\infty} \frac{(-i)}{\pi k} e^{-ikx}} = \frac{1}{2} + \sum_{\substack{k=1 \\ k \text{ odd}}}^{\infty} \frac{(-i)}{\pi k} \underbrace{\left(e^{ikx} - e^{-ikx} \right)}_{\substack{i \sin kx - (-i \sin kx) \\ = 2i \sin kx}},$$

i.e., $F_f(x) = \frac{1}{2} + \sum_{\substack{k=1 \\ k \text{ odd}}}^{\infty} \frac{2}{\pi k} \sin kx$.



Here, e.g., we see that $F_f(\pi) = \frac{1}{2} \neq f(\pi)$, so we have neither pointwise nor uniform convergence. But it looks like some type of convergence should hold.

Example B: $f(x) = (x-\pi)^2$ on $[0, 2\pi]$



A computation (see HW) shows $F_f(x) = \frac{\pi^2}{3} + \sum_{k \neq 0} \frac{2}{k^2} e^{ikx} = \frac{\pi^2}{3} + \sum_{k=1}^{\infty} \frac{4}{k^2} \cos kx$,

which converges uniformly (according to the Weierstrass M-test), i.e., $F_f(x) = f(x)$.

As a corollary we find $\sum_{k=1}^{\infty} \frac{4}{k^2} = f(0) - \frac{\pi^2}{3} = \pi^2 - \frac{\pi^2}{3} = \frac{2\pi^2}{3}$, i.e., $\sum_{k=1}^{\infty} \frac{1}{k^2} = \frac{\pi^2}{6}$.

Question: What is the right kind of convergence for functions as in Example A?

Answer: Convergence in the norm coming from our inner product.

First, note that

$$\|f - \sum_{k=-n}^{n} \hat{f}_k e_k\|^2 = \langle f - \sum_{k=-n}^{n} \hat{f}_k e_k, f - \sum_{k=-n}^{n} \hat{f}_k e_k \rangle$$

$$= \|f\|^2 - \sum_{k=-n}^{n} \Big(\underbrace{\langle f, \hat{f}_k e_k \rangle}_{\substack{= \hat{f}_k \langle f, e_k \rangle \\ = \hat{f}_k \overline{\hat{f}(k)} = |\hat{f}_k|^2}} + \underbrace{\langle \hat{f}_k e_k, f \rangle}_{= \hat{f}_k \overline{\hat{f}_k} = |\hat{f}_k|^2} \Big) + \sum_{k=-n}^{n} \sum_{j=-n}^{n} \underbrace{\langle \hat{f}_j e_j, \hat{f}_k e_k \rangle}_{= \hat{f}_j \overline{\hat{f}_k} \delta_{jk}}$$

$$\Rightarrow \|f - \sum_{k=-n}^{n} \hat{f}_k e_k\|^2 = \|f\|^2 - \sum_{k=-n}^{n} |\hat{f}_k|^2$$

As a corollary, we get Bessel's inequality $\sum_{k=-\infty}^{\infty} |\hat{f}_k|^2 \le \|f\|^2 = \frac{1}{2\pi}\int_0^{2\pi} |f(x)|^2\,dx$.

Furthermore: $\underbrace{\|f - \sum_{k=-n}^{n} \hat{f}_k e_k\| \xrightarrow{n\to\infty} 0}_{\text{called "mean-square convergence"}} \iff \|f\|^2 = \sum_{k=-\infty}^{\infty} |\hat{f}_k|^2$ (Parseval identity)

For Example A we find $\|f\|^2 = \frac{1}{2\pi}\int_0^a dx = \frac{a}{2\pi}$ and

$$\sum_{k=-\infty}^{\infty} |\hat{f}_k|^2 = \underbrace{\left(\frac{a}{2\pi}\right)^2}_{=|\hat{f}_0|^2} + \sum_{k\neq 0} \Big|\underbrace{\frac{i}{2\pi k}\left(e^{-ika} - 1\right)}_{\hat{f}_k}\Big|^2$$

$= \ldots$ (see HW; use results from Ex. B)

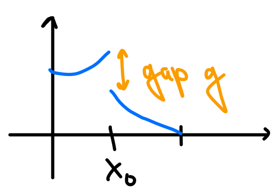
$= \frac{a}{2\pi}$, i.e., the Fourier series converges to f in mean-square.

In general, we can approximate any Riemann-integrable f by such square pulses, which leads to the following result:

Theorem: Let $f: [0, 2\pi] \to \mathbb{C}$, $f(0) = f(2\pi)$ be Riemann-integrable. Then $\|f - \sum_{k=-n}^{n} \hat{f}_k e^{ikx}\| \xrightarrow{n\to\infty} 0$, i.e., $\sum_{k=-\infty}^{\infty} \hat{f}_k e^{ikx} \to f(x)$ in mean-square.

(We omit the proof here.)

Let us mention two more properties of the Fourier series. Suppose f is piece-wise continuous and piece-wise differentiable but has a discontinuity at x_0:



Then:

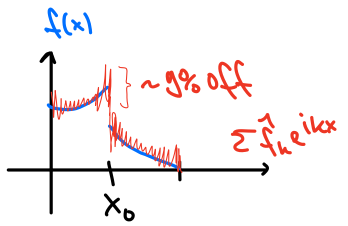
- $\sum_{k=-n}^{n} \hat{f}_k e^{ikx} \xrightarrow{n\to\infty} \frac{1}{2}\Big(\underbrace{\lim_{x\downarrow 0} f(x)}_{=:f(x_0^+)} + \underbrace{\lim_{x\uparrow 0} f(x)}_{=:f(x_0^-)} \Big)$ (as we saw in Example A)

- Let $g := f(x_0^+) - f(x_0^-)$ be the gap at the discontinuity.

Then $\sum_{k=-n}^{n} \hat{f}_k e^{ikx}\Big|_{x=x_0+\frac{\pi}{n}} \xrightarrow{n\to\infty} f(x_0^+) + gc$, with $c \approx 0.089\ldots$

and $\sum_{k=-n}^{n} \hat{f}_k e^{ikx}\Big|_{x=x_0-\frac{\pi}{n}} \xrightarrow{n\to\infty} f(x_0^-) - gc$.

$\Rightarrow$ Near a discontinuity, the Fourier series is ~9% off. This is called "Gibbs phenomenon".



We can check this directly for Example A with $a=\pi$: $(f(x_0^+)=0,\ g=-1)$

$$\sum_{k=-n}^{n} \hat{f}_k e^{ikx}\Big|_{x=x_0+\frac{\pi}{n}} = \frac{1}{2} + \sum_{\substack{k=1\\ k\,\text{odd}}}^{n} \frac{2}{\pi k} \sin\left(kx_0 + k\frac{\pi}{n}\right)$$

$x_0=\pi$

$$= \frac{1}{2} + \sum_{\substack{k=1\\ k\,\text{odd}}}^{n} \frac{2}{\pi k} \underbrace{\sin\left(\pi k + \pi \frac{k}{n}\right)}_{= -\sin\left(\pi\frac{k}{n}\right)}$$

$$= \frac{1}{2} - \underbrace{\sum_{\substack{k=1\\ k\,\text{odd}}}^{n} \frac{1}{\left(\frac{n}{2}\right)} \frac{\sin\left(\pi\frac{k}{n}\right)}{\pi\left(\frac{k}{n}\right)}}_{\xrightarrow{n\to\infty} \int_0^1 dx \frac{\sin(\pi x)}{\pi x} = \frac{1}{\pi}\int_0^{\pi} dy \frac{\sin y}{y}}$$

$\pi x = y$ substitution

$$\xrightarrow{n\to\infty} \frac{1}{2} - \frac{1}{\pi}\int_0^{\pi} dy \frac{\sin y}{y} \approx -0.089\ldots \quad \checkmark$$