5. Complex Analysis

In this chapter we consider functions $f: D \to \mathbb{C}$, where $D \subset \mathbb{C}$ is a domain. This will lead to a deeper understanding of functions such as exp and log (hence the German name "Funktionentheorie") and also useful tools, e.g., for integration.

Differentiability is defined in the usual way:

A fct. $f: D \to \mathbb{C}$ ($D \subset \mathbb{C}$ a domain) is differentiable at $z_0 \in D$ if

$$f(z) = f(z_0) + c(z-z_0) + |z-z_0| h(z) \quad \text{for some } c \in \mathbb{C} \text{ and with } \lim_{z\to z_0} h(z) \to 0.$$

Alternatively, we can identify $\mathbb{C}$ with $\mathbb{R}^2$ by writing $z = x+iy = \binom{x}{y}$ and $f = u+iv = \binom{u}{v}$.

$f: D \subset \mathbb{R}^2 \to \mathbb{R}^2, \binom{x}{y} \mapsto \binom{u(x,y)}{v(x,y)}$ is differentiable at $z_0 = x_0 + iy_0$ if

$$f(z) = f(z_0) + A\cdot(z-z_0) + |z-z_0|\, h(z) \quad \text{for some real } 2\times 2 \text{ matrix } A, \text{ and } \lim_{z\to z_0} h(z) = \binom{0}{0}.$$

If f is differentiable, then $A = \begin{pmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{pmatrix}$.

Now we can compare both linear expressions (let us set $z_0 = 0$ here for simplicity):

$$cz = (a+ib)(x+iy) = (ax-by) + i(bx+ay) = \begin{pmatrix} ax-by \\ bx+ay \end{pmatrix} = \begin{pmatrix} a & -b \\ b & a \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} \overset{!}{=} A \begin{pmatrix} x \\ y \end{pmatrix}.$$

Comparing with A yields: $\boxed{\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \text{ and } \frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y} \text{, the Cauchy-Riemann equations.}}$

("C-R eq.s")

We have proven

Theorem: A function $f: D \to \mathbb{C}$, $D \subset \mathbb{C}$ a domain, is complex differentiable ("holomorphic") at $z_0 \in D$ if and only if f is real differentiable and the C-R eq.s hold at z_0.

Remark: The C-R eq.s have many interesting consequences and make the theory of holomorphic fct.s very rich. E.g.:
- Every holomorphic fct. is arbitrarily often differentiable.
- Every holomorphic fct. has a power series expansion.
- Every fct. holomorphic in all of $\mathbb{C}$ and bounded must be constant.

Examples:

- $f(z) = z^2 = (x+iy)^2 = \underbrace{x^2 - y^2}_{u} + i\underbrace{2xy}_{v}$

$$\Rightarrow \frac{\partial u}{\partial x} = 2x = \frac{\partial v}{\partial y} \checkmark \qquad \frac{\partial u}{\partial y} = -2y = -\frac{\partial v}{\partial x} \checkmark \Rightarrow f(z) = z^2 \text{ is holomorphic}$$

- $f(z) = \bar{z} = x - iy$

$$\Rightarrow \frac{\partial u}{\partial x} = 1 \neq \frac{\partial v}{\partial y} = -1 \Rightarrow f(z) = \bar{z} \text{ is not holomorphic}$$

- One can show that any polynomial and $\exp(z)$ (and thus $\sin z, \cos z$) are holomorphic.

Next, consider complex line integrals. Let γ be a curve in D. Then

$$\int_\gamma f(z)\,dz = \int_\gamma (u+iv)(dx+idy) = \int_\gamma (u\,dx - v\,dy) + i\int_\gamma (v\,dx + u\,dy)$$

$$= \int_\gamma \begin{pmatrix} u \\ -v \end{pmatrix} \cdot \begin{pmatrix} dx \\ dy \end{pmatrix} + i \int_\gamma \begin{pmatrix} v \\ u \end{pmatrix} \cdot \begin{pmatrix} dx \\ dy \end{pmatrix}.$$

Now suppose γ is simple and closed, and $\gamma = \partial S$ for some $S \subset D$.

$$\text{Then } \int_\gamma f(z)\,dz \overset{\text{Green's thm.}}{=} \int_S \underbrace{\nabla^\perp \cdot \begin{pmatrix} u \\ -v \end{pmatrix}}_{} dx\,dy + i \int_S \underbrace{\nabla^\perp \cdot \begin{pmatrix} v \\ u \end{pmatrix}}_{} dx\,dy = 0$$

$$= \begin{pmatrix} -\frac{\partial}{\partial y} \\ \frac{\partial}{\partial x} \end{pmatrix} \cdot \begin{pmatrix} u \\ -v \end{pmatrix} = -\frac{\partial u}{\partial y} - \frac{\partial v}{\partial x} = 0$$

C-R eq.s

$$= \begin{pmatrix} -\frac{\partial}{\partial y} \\ \frac{\partial}{\partial x} \end{pmatrix} \cdot \begin{pmatrix} v \\ u \end{pmatrix} = -\frac{\partial v}{\partial y} + \frac{\partial u}{\partial x} = 0$$

C-R eq.s

We have proven $\boxed{\text{Cauchy's integral theorem:}}$

This is the most general condition for Green's thm. to hold.

Theorem: Let $f: D \to \mathbb{C}$, $D \subset \mathbb{C}$ a simply connected domain, be holomorphic, and $\gamma \subset D$ a closed curve. Then $\int_\gamma f(z)\,dz = 0$.

From this, much of the theory of complex analysis will follow.