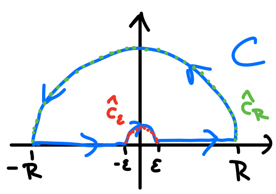

With Cauchy's thm. we can already compute some integrals, e.g., the Dirichlet integral (see "Feynman's trick" from Session 15):

Let $A_R = \int_{-R}^{R} \frac{e^{ix}-1}{x}\,dx \Rightarrow \operatorname{Re} A = \underbrace{\int_{-R}^{R} \frac{\cos x - 1}{x}\,dx}_{=0}, \ \operatorname{Im} A = \int_{-R}^{R} \frac{\sin x}{x}\,dx$

$$\Rightarrow A_R = i \operatorname{Im} A_R \Rightarrow \int_{-\infty}^{\infty} \frac{e^{ix}-1}{x}\,dx = i \int_{-\infty}^{\infty} \frac{\sin x}{x}\,dx = i2 \int_{0}^{\infty} \frac{\sin x}{x}\,dx$$

$$\Rightarrow \int_0^\infty \frac{\sin x}{x}\,dx = \frac{1}{2i} \int_{-\infty}^{\infty} \underbrace{\frac{e^{ix}-1}{x}}_{f(x)}\,dx$$



$$\int_C f(z)\,dz = \int_{-R}^{-\varepsilon} f(z)\,dz + \underbrace{\int_{\hat{C}_\varepsilon} f(z)\,dz}_{} + \int_{\varepsilon}^{R} f(z)\,dz + \underbrace{\int_{\hat{C}_R} f(z)\,dz}_{} = 0$$

$$\int_{\hat{C}_\varepsilon} f(z)\,dz = \int_\pi^0 f(\varepsilon e^{it})\,\varepsilon i e^{it}\,dt = \int_\pi^0 \frac{e^{i\varepsilon e^{it}}-1}{\varepsilon e^{it}}\,\varepsilon i e^{it}\,dt \le \int_\pi^0 (e^{\varepsilon}-1)\,dt \le \pi(e^\varepsilon - 1) \xrightarrow{\varepsilon\to 0} 0$$

$$\int_{\hat{C}_R} f(z)\,dz = \int_0^\pi f(Re^{it})\,Rie^{it}\,dt = \int_0^\pi \frac{e^{iR(\cos t + i\sin t)}-1}{Re^{it}}\,(Rie^{it})\,dt = i\int_0^\pi \Big(\underbrace{e^{iR\cos t}\, e^{-R\sin t}}_{\to 0 \text{ as } R\to\infty\ \forall t\in(0,\pi)} - 1\Big)\,dt \xrightarrow{R\to\infty} -i\pi$$

$$\Rightarrow \int_{-\infty}^{\infty} f(x)\,dx = i\pi \Rightarrow \int_0^\infty \frac{\sin x}{x}\,dx = \frac{\pi}{2}$$

Cauchy's integral theorem implies Cauchy's integral formula:

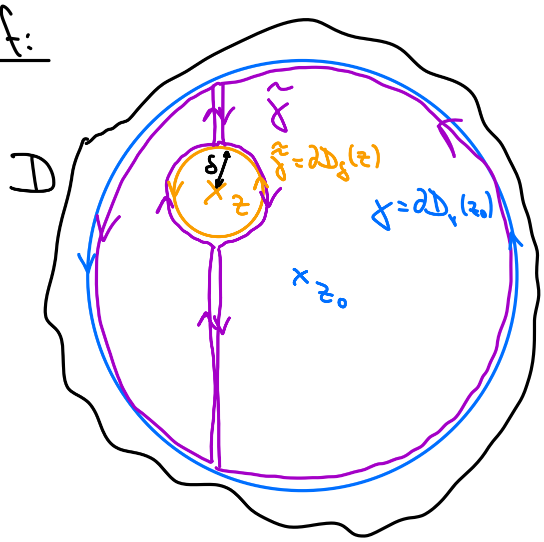
Corollary: Let $f: D \to \mathbb{C}$, $D \subset \mathbb{C}$ a domain, be holomorphic. Then

$$f(z) = \frac{1}{2\pi i} \int_{\partial D_r(z_0)} \frac{f(w)}{w-z}\, dw \quad \text{for any } z \in \underbrace{D_r(z_0)}_{\text{disc with radius } r \text{ around } z_0} := \{ w \in \mathbb{C} : |z_0 - w| < r \} \subset D.$$

↑ counterclockwise

$\Rightarrow$ The values of f in a disc are determined only by the values on the boundary of the disc.

Proof:



We have: $\displaystyle\int_\gamma \frac{f(w)}{w-z}\, dw = \underbrace{\int_{\tilde{\gamma}} \frac{f(w)}{w-z}\, dw + \int_{\tilde{\tilde{\gamma}}} \frac{f(w)}{w-z}\, dw}$

$$= \int_{\tilde{\gamma}_R} \frac{f(w)}{w-z}\, dw + \int_{\tilde{\gamma}_L} \frac{f(w)}{w-z}\, dw \overset{\text{by Cauchy's int. thm.}}{=} 0.$$

holomorphic in the enclosed areas (which are simply connected, even star-shaped for δ small enough)

$$\Rightarrow \text{For any } \delta > 0: \int_{\partial D_r(z_0)} \frac{f(w)}{w-z}\, dw = \int_{\partial D_\delta(z)} \frac{f(w)}{w-z}\, dw \overset{\text{change of variables } w-z \to w}{=} \frac{1}{2\pi i} \int_{\partial D_\delta(0)} \frac{f(z+w)}{w}\, dw$$

$$= \frac{1}{2\pi i} \int_0^{2\pi} \frac{f(z+\gamma(t))}{\gamma(t)} \gamma'(t)\, dt \overset{\gamma: [0,2\pi] \to \mathbb{C},\ \gamma(t) = \delta e^{it}}{=} \frac{1}{2\pi i} \int_0^{2\pi} \frac{f(z+\delta e^{it})}{\delta e^{it}}\, i\delta e^{it}\, dt = \frac{1}{2\pi} \int_0^{2\pi} f(z+\delta e^{it})\, dt$$

By uniform convergence! (since f holomorphic) $\longrightarrow$ $\xrightarrow{\delta \to 0} f(z)$. $\square$

Note: the proof clearly works as well if we replace $\partial D_r(z_0)$ by any closed simple curve γ encircling z.

It follows:

$\boxed{\text{\underline{Corollary:} Under the same assumptions as above, } f^{(n)}(z) = \frac{n!}{2\pi i} \int_\gamma \frac{f(w)}{(w-z)^{n+1}}\, dw.}$

And:

$\boxed{\text{\underline{Corollary:} If } f: D \to \mathbb{C},\ D \subset \mathbb{C} \text{ a domain, is holomorphic, then } f \in C^\infty \text{ and furthermore } f \text{ is analytic, i.e., it has a convergent Taylor series.}}$

__Proofs:__ $\frac{1}{w-z} = \frac{1}{w-z_0+z_0-z} = \frac{1}{w-z_0} \frac{1}{1-\frac{z-z_0}{w-z_0}} = \frac{1}{w-z_0} \sum_{k=0}^{\infty} \left(\frac{z-z_0}{w-z_0}\right)^k$ for $\left|\frac{z-z_0}{w-z_0}\right| < 1$,

geometric series

with uniform convergence!

$$\Rightarrow f(z) = \frac{1}{2\pi i} \int_\gamma \frac{f(w)}{w-z}\, dw = \sum_{k=0}^{\infty} (z-z_0)^k \underbrace{\frac{1}{2\pi i} \int_\gamma \frac{f(w)}{(w-z_0)^{k+1}}\, dw}_{= \frac{f^{(k)}(z_0)}{k!}} = \sum_{k=0}^{\infty} \frac{f^{(k)}(z_0)}{k!} (z-z_0)^k. \qquad \square$$

take sum out of integral by uniform convergence

Next: More generally, one can write down a $\boxed{\text{Laurent series } \sum_{k=-\infty}^{\infty} a_k (z-z_0)^k.}$

It converges if
- $|z-z_0| < R$ (= radius of convergence of $\sum_{k=0}^{\infty} a_k (z-z_0)^k$),
- and $\frac{1}{|z-z_0|} < \tilde{R}$ (= radius of convergence of $\sum_{k=-\infty}^{-1} a_k (z-z_0)^k$),

i.e., it converges on an annulus $\Omega = \{ z \in \mathbb{C} : \frac{1}{\tilde{R}} < |z-z_0| < R \}$.

Indeed, one can show that any holomorphic function $f: \Omega \to \mathbb{C}$ has a Laurent series.

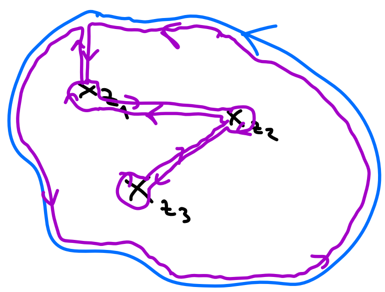
An interesting case is when $\tilde{R} = \infty$, i.e., $0 < |z-z_0| < R$. Such points z_0 are called $\boxed{\text{isolated singularities.}}$

For these, we get:

$$\int_{\partial D_r(z_0)} f(z)\,dz = \sum_{k=-\infty}^{\infty} a_k \underbrace{\int_{\partial D_r(z_0)} (z-z_0)^k\,dz}_{= \int_0^{2\pi} (re^{it})^k\, ire^{it}\,dt \,=\, ir^{k+1}\int_0^{2\pi} e^{i(k+1)t}\,dt \,=\, 2\pi i r^{k+1}\delta_{k,-1}} = 2\pi i\, a_{-1}.$$

We call $a_{-1} := \mathrm{Res}(f, z_0)$ the residue of f at z_0.

Generalizing along the picture



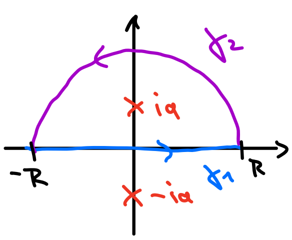
yields:

Theorem (Residue Theorem): Let $D \subset \mathbb{C}$ be a simply connected domain. Let $f: D \to \mathbb{C}$ be holomorphic except at a finite number of isolated points $z_1, \dots, z_n$. Let γ be a simple closed curve enclosing all $z_1, \dots, z_n$. Then

$$\boxed{\int_\gamma f(z)\,dz = 2\pi i \sum_{k=1}^{n} \mathrm{Res}(f, z_n).}$$

Application: Compute $\displaystyle\int_{-\infty}^{\infty} \frac{1}{(x^2+a^2)^4}\,dx$, $0 \neq a \in \mathbb{R}$.

We def. $f(z) = \frac{1}{(z^2+a^2)^4}$. It has two isolated singularities at $\pm ia$.



The residue thm. tells us that for R large enough

$$\underbrace{\int_{\gamma_1} f(z)\,dz}_{= \int_{-R}^{R} \frac{1}{(x^2+a^2)^4}\,dx} + \underbrace{\int_{\gamma_2} f(z)\,dz}_{= \int_0^{\pi} (R^2e^{2it}+a^2)^{-4}\, iRe^{it}\,dt} = \int_{\gamma=\gamma_1+\gamma_2} f(z)\,dz = 2\pi i\,\mathrm{Res}(f, ia).$$

Note that $\left| \int_{\gamma_2} f(z)\,dz \right| \le \int_0^\pi |R^2 e^{2it} + a^2|^{-4} R\,dt \le \frac{R}{|R^2-a^2|^4} \underbrace{\int_0^\pi dt}_{=\pi} \xrightarrow{R\to\infty} 0.$

Thus, we just need to compute $\mathrm{Res}(f, ia)$. Let us write $z = ia + w$ and

$$f(z) = \left((ia+w)^2 + a^2 \right)^{-4} = \left(-a^2 + 2iaw + w^2 + a^2 \right)^{-4} = w^{-4} (2ia+w)^{-4}$$

$$= (2iaw)^{-4} \underbrace{\left(1 + \frac{w}{2ia} \right)^{-4}}_{= \sum_{k=0}^{\infty} \binom{-4}{k} \left(\frac{w}{2ia} \right)^k}$$

$\mathrm{Res}(f, ia)$ is the coefficient with power w^{-1}, i.e. $k=3$. Then

$$\mathrm{Res}(f, ia) = (2ia)^{-4} \binom{-4}{3} \left(\frac{1}{2ia} \right)^3 = (2ia)^{-7} \frac{(-4)(-4-1)(-4-2)}{3!} = \ldots = -i \frac{5}{32} a^7.$$

$$\Longrightarrow \int_{-\infty}^{\infty} \frac{1}{(x^2+a^2)^4}\,dx = \lim_{R\to\infty} \int_{-R}^{R} \frac{1}{(x^2+a^2)^4}\,dx = 2\pi i \left(-i \frac{5}{32} a^7 \right) = \pi \frac{5}{16} a^7.$$