Constructor University | February 14, 2023
Spring 2023

Topology and Manifolds

Homework 2

Due on February 21, 2023, before the tutorial

Problem 1 [**4 points**]

Let $f : X \to Y$ be a map between topological spaces X and Y with the property that every $p \in X$ has a neighborhood U such that $f|_U$ is continuous. Prove that then f is continuous.

Problem 2 [**4 points**]

Prove that every continuous bijection $f : X \to Y$ with X compact and Y Hausdorff is a homeomorphism. (*Hint: In Y, every compact set is closed (why?); in X, every closed subset is compact (why?). Then note that f takes closed sets to closed sets (why?).*)

Problem 3 [**4 points**]

Prove that for a topological space (X, τ), path connectedness implies connectedness.

Problem 4 [**4 points**]

Let $U \subset \mathbb{R}^n$ be open and connected. Prove that if the derivative of $f : U \to \mathbb{R}$ is zero everywhere in U, then f is constant. (*Hint: Use the previous homework sheet and connect some given point to any other point by a finite sequence of line segments.*) If U is not connected, give a counterexample of an f with zero derivative that is not constant.

Problem 5 [**4 points**]

We consider the topologist's sine curve, i.e., the set

$$A = \left\{ \left(x, \sin\frac{1}{x} \right) \in \mathbb{R}^2 : x \in (0, 1] \right\} \cup \{(0, 0)\}$$

with the subspace topology. Prove that A is connected but not path-connected.