Constructor University — March 16, 2023
Spring 2023

Topology and Manifolds

Homework 6

Due on March 23, 2023, before class

Problem 1 [**5 points**]

In class we discussed the following lemma: Let M be a compact m-dimensional smooth manifold, $F : M \to \mathbb{R}^N$ a smooth injective immersion, and $N > 2m + 1$. Then there is a dense set of vectors $v \in \mathbb{S}^{N-1}$ such that $\pi_v \circ F$ is a smooth injective immersion, where $\pi_v(x) = x - \langle x, v\rangle v$ is the projection orthogonal to $v \in \mathbb{S}^{N-1}$. Finish the proof from class by showing that $\pi_v \circ F$ is indeed an immersion.

Problem 2 [**5 points**]

Finish the proof of the Whitney Embedding Theorem (compact case) from class by showing that the map F that was defined is indeed injective and an immersion.

Problem 3 [**5 points**]

Let X and Y be topological spaces and $f : X \to Y$ a continuous map. The map f is called *proper* if for every compact set $K \subset Y$ the preimage $f^{-1}(K)$ is also compact.

(a) Give an example of a proper and an example of a non-proper continuous function $f : \mathbb{R} \to \mathbb{R}$.

(b) Suppose X is compact and Y is Hausdorff. Show that then $f : X \to Y$ is proper.

(c) Suppose $f : X \to Y$ is a bijective proper continuous map. Show that then f is a homeomorphism.

Problem 4 [**5 points**]

(a) Consider the standard embedding $S^1 \subset \mathbb{R}^2$. For each point $p = (x, y) \in S^1$, define $X(p) = (-y, x)$. Show that $X(p)$ is a basis for T_pS^1 for all $p \in S^1$.

(b) Consider the standard embedding $S^3 \subset \mathbb{R}^4$. For each point $p = (x, y, z, t) \in S^3$, define

$$\begin{aligned} X_1(p) &= (-y, x, t, -z) \\ X_2(p) &= (-z, -t, x, y) \\ X_3(p) &= (-t, z, -y, x). \end{aligned}$$

Show that $X_1(p), X_2(p), X_3(p)$ form a basis of T_pS^3 for all $p \in S^3$.