Constructor University
Spring 2023

March 23, 2023

Topology and Manifolds

Homework 7

Due on March 30, 2023, before class

Problem 1 [**4 points**]

Prove that every Lie group homomorphism has constant rank.

Problem 2 [**4 points**]

Let G^0 be the identity component of a Lie group G. Prove that G^0 is a Lie group, $\dim G^0 = \dim G$, and that G^0 is a normal subgroup of G.

Problem 3 [**4 points**]

We consider the group $O_n(\mathbb{R})$ of orthogonal real $n \times n$ matrices (i.e., matrices whose columns and rows are orthonormal).

(a) Prove that $O_n(\mathbb{R})$ is a compact group. *(Hint: Consider the map $g \mapsto g^T g$, where g^T is the transpose of g.)*

(b) Show that the image of the homomorphism $\det : O_n(\mathbb{R}) \to \mathbb{R}^*$ is the subgroup $\{-1, +1\}$.

(c) Now consider the special orthogonal group $SO_n(\mathbb{R})$, i.e., the subgroup of $O_n(\mathbb{R})$ consisting of matrices g with $\det g = 1$. Show that $SO_2(\mathbb{R})$ is isomorphic to the circle S^1, and thus abelian. Is $O_2(\mathbb{R})$ also abelian?

(d) Prove that $SO_3(\mathbb{R})$ is not abelian. *(Hint: Consider rotations with different axes.)*

Problem 4 [**4 points**]

(a) Prove the local coordinate formula for Lie brackets that was stated in class, i.e.,

$$[X, Y] = \sum_{i,j=1}^{n} \left(X^i \frac{\partial Y^j}{\partial x^i} - Y^i \frac{\partial X^j}{\partial x^i} \right) \frac{\partial}{\partial x^j}.$$

(b) Compute the Lie bracket explicitly for the following vector fields in $\mathbb{R}^3$:

$$X = x\frac{\partial}{\partial x} + \frac{\partial}{\partial y} + yz\frac{\partial}{\partial z}$$
$$Y = y\frac{\partial}{\partial x} + x(z+1)\frac{\partial}{\partial z}.$$

Problem 5 [4 points]

Prove the properties a) - e) of the Lie bracket that we discussed in class, i.e., bilinearity, antisymmetry, the Jacobi identity, behavior under multiplication with C^∞ functions, and behavior under pushforwards.